The expected long-time behavior of a solution of the spatially homogeneous Boltzmann equation seems to leave little room for imagination: if the initial datum has finite kinetic energy, then as time t goes to $+\infty$ the solution should converge to a Maxwellian distribution. In 1997-1998 I thought about two related, but seemingly more original, problems. One was the possibility to keep the energy finite, but let time go to $-\infty$ instead of $+\infty$; then, the asymptotic behavior looks a priori unclear, but what is more, there is good reason to suspect that there is no solution at all. The other was to relax the assumption of finite energy, and try to construct self-similar solutions which would capture the asymptotic behavior of solutions with infinite energy, and would play the role of the stable stationary laws in classical probability theory. In a preliminary investigation, it looked very reasonable to consider these problems in the simple setting of the spatially homogeneous Boltzmann equation with Maxwellian collision kernel.

On the first topic I made some progress, although far from decisive. I wrote down the text below and added it to my PhD (June 1998) as an appendix (here I only changed the references). On the second topic I made no progress, and in fact began to suspect that those self-similar solutions did not exist. In 2001, Bobylev and Cercignani proved that I was wrong, by exhibiting such self-similar solutions, constructed with the help of Fourier transform. They also proved that for $t \to -\infty$ there exists no solution with finite moments of all order: this is a weakened version of the conjecture explained in the following text (the full version of the conjecture would be that just second moment is sufficient). Their paper, which since then appeared in the Journal of Statistical Physics, *may be consulted for more information.*

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IS THERE ANY BACKWARD SOLUTION OF THE BOLTZMANN EQUATION ?

In Chapter XII of their famous book [5], Truesdell and Muncaster consider a spatially homogeneous gas of Maxwellian molecules, and prove that all the moments of order 2 and 3, namely $\int f v_i v_j$, $\int f v_i v_j v_k$, converge to their equilibrium values exponentially fast, with a known relaxation constant. They add (p.191): "In a much more concrete way than Boltzmann's H-theorem, [these quantities] illustrate the *irreversibility* of the behavior of the kinetic gas. This irreversibility is particularly striking if we attempt to trace the origin of a grossly homogeneous condition by considering past times instead of future ones. Indeed *the magnitude of each component of [the pressure tensor] and [the tensor of the moments of order 3] that is not 0 at* $t = 0$ *tends to* ∞ *as* $t \to -\infty$. Thus any present departure from kinetic equilibrium must be the outcome of still greater departure in the past."

Appealing as this image may be, it is our conviction that it is actually *impossible* to let $t \to -\infty$ (which is maybe an even more striking manifestation of irreversibility ?) More precisely, we state the following

Conjecture 1. *Let $f(t,v)$ be a solution of the Boltzmann equation with Maxwellian molecules, with finite mass and energy, which is defined on all $\mathbb{R}\times\mathbb{R}^N$. Then f is stationary : for all time t, $f(t,v)=M(v)$, where M is the Maxwellian distribution with same mass, momentum and energy as f.*

This problem may seem academic, but we shall point out that it can be seen as intimately connected with the important problem of the uniformity of the trend to equilibrium. In addition, we shall give a proof that Conjecture 1 is true for the Landau equation, precisely because for this problem *the tails of distribution do not bother the trend to equilibrium.*

Let us comment on the positivity condition in Conjecture 1. The classical theorems of existence of a solution to the Boltzmann equation in small times do not mind which direction of the time is considered. But the positivity is preserved only when time goes forward, and not backward. As mentioned by Bobylev [1], it is possible to construct initial datum that are partially negative, and such that the corresponding solutions to the Boltzmann equation blow up in finite time. In fact, the positivity is essential for the mathematical estimates as well as the physical meaning.

Before we go further, it may be enlightening to treat the case of the heat equation $\partial_t f = \Delta f$. For this equation, it is easy to construct (explicitly) solutions that exist for all times (following Cabannes, we shall call them "eternal"). However, *they are never nonnegative*, except for the trivial case $f=0$. This is immediate in the case when the energy of f is finite : then it grows linearly in time, with a speed equal to the total mass of f, and therefore must be negative at some time. In the case when the energy is infinite, Conjecture 1 also holds, by the following argument (communicated to us by S. Poirier) :

Proposition 1. *(i) There exists $K<1$ such that the following property holds. Let f be a solution of the heat equation on the interval of time $[-T,0]$. Then the ball $(|v|\le T)$ contains at most a proportion K of the total mass of $f(0,\cdot)$.*
(ii) As a consequence, if f is an eternal solution of the heat equation, then $f(0,\cdot)$ is not integrable.

Proof. It is clear that (i) implies (ii). To prove (i), we set

$$K=\frac{\int_{|v|\le 1} e^{-\frac{|v|^2}{4}}\,dv}{\int_{\mathbb{R}^N} e^{-\frac{|v|^2}{4}}\,dv}=\frac{\int_{|v|\le\sqrt{T}} e^{-\frac{|v|^2}{4T}}\,dv}{\int_{\mathbb{R}^N} e^{-\frac{|v|^2}{4T}}\,dv}<1.$$

Then, let us write

$$f(0)=g*\frac{e^{-\frac{|v|^2}{4T}}}{(4\pi T)^{N/2}}$$

for some function $g\ge 0$. Then,

$$\int_{|v|\le\sqrt{T}} f=\int_{|v|\le\sqrt{T}} dv\int dw\, g(w)\frac{e^{-\frac{|v-w|^2}{4T}}}{(4\pi T)^{N/2}}$$

$$=\frac{1}{(4\pi T)^{N/2}}\int dw\, g(w)\int_{|v|\le\sqrt{T}} dv\, e^{-\frac{|v-w|^2}{4T}}.$$

Since $e^{-|x|^2}$ is a decreasing function of $|x|$,

$$\int_{|v|\le\sqrt{T}} dv\, e^{-\frac{|v-w|^2}{4T}} = \int_{|v+w|\le\sqrt{T}} dv\, e^{-\frac{|v|^2}{4T}}$$
$$\le \int_{|v|\le\sqrt{T}} dv\, e^{-\frac{|v|^2}{4T}} \le K\int_{\mathbb{R}^N} dv\, e^{-\frac{|v|^2}{4T}}.$$

Hence,

$$\int_{|v|\le\sqrt{T}} f \le K\int_{\mathbb{R}^N} dw\, g(w)\int_{\mathbb{R}^N} dv\, \frac{e^{-\frac{|v|^2}{4T}}}{(4\pi T)^{N/2}} = K\int_{\mathbb{R}^N} g(w)\, dw.$$

□

This proof is an illustration of our general strategy : the impossibility of solving the backward equation can be seen as a consequence of the uniformity of the trend towards "equilibrium" (here, 0).

Let us now prove that, as far as the pressure deviator is concerned, there can be no departure from equilibrium for eternal solutions of the Boltzmann (or Landau) equation.

Proposition 2. *Let f be an eternal solution of the Boltzmann (or Landau) equation with Maxwellian molecules. Then all the second order moments $\int f v_i v_j$ are always equal to their equilibrium values.*

Proof. We treat the case of the Landau equation, which is exactly similar to that of the Boltzmann equation. From the study in [7], we deduce that (noting M the Maxwellian equilibrium associated to f)

$$\int f(0,v)v_iv_j\, dv - \int M v_iv_j\, dv = e^{-\lambda T}\int \Big(f(-T,v) - M(v)\Big)v_iv_j\, dv$$

for some constant $\lambda > 0$ depending only on the mass and energy of f. Hence

$$\left|\int f(0,v)v_iv_j\, dv - \int M(v)v_iv_j\, dv\right| \le 4Ee^{-\lambda T},$$

where E is the energy of f. Letting T go to $+\infty$, we get the result. □

Sketch of proof of Conjecture 1. Let f be an eternal solution of the Boltzmann (or Landau) equation, and let t_n be any sequence of times going to $+\infty$. We assume without loss of generality that f is a centered probability distribution with energy $N/2$. We set

$$f^n(t,v) = f(t-t_n, v).$$

Since (f^n) satisfies a uniform estimate for mass and energy (of course, not for the entropy!), we know that up to extraction, (f^n) converges, weakly in measure sense on all finite time-interval, towards a measure $\mu(t,v)$ with finite mass and energy. Moreover, using $ff_* \rightharpoonup \mu\mu_*$ in $M^1(\mathbb{R}^N\times\mathbb{R}^N_*)$, it is easy to pass to the limit in the weak formulation of the Boltzmann equation (Cf. [6]), and therefore μ is a weak solution of the Boltzmann equation (in particular, the energy of μ is preserved with time).

Now, let us prove that $\mu(t,\cdot)\rightharpoonup m$ as $t\to\infty$, where m is the Maxwellian distribution with the same moments as μ (note that the energy of μ may be less than the energy of f!!). This was in fact proven by Gabetta, Wennberg and Toscani in [3], but we shall give here a simple and self-contained proof which will show once again the interest of Bobylev's lemma. Since the Boltzmann equation commutes with the convolution by any Maxwellian M_δ, $\mu * M_\delta$ is still a solution of the Boltzmann equation (in fact, one has to check that Bobylev's lemma remains true in weak formulation, which is not difficult), but it is C^∞, and hence it is a strong solution with finite entropy. Therefore, it converges strongly towards $M * M_\delta = M_{1+\delta}$ as $t \to \infty$. Hence, for all ξ, $\widehat{\mu}(t,\xi)\widehat{M_\delta}(\xi) \to \widehat{m}(\xi)\widehat{M_\delta}(\xi)$, and of course $\widehat{\mu}(t,\xi) \to \widehat{m}(\xi)$. This entails that $\mu \to M$ weakly in measure sense.

Now, we consider a distance d which is nonexpansive for the Boltzmann semigroup : as we saw in [4], the Tanaka-Wasserstein distance, or the distance d_2, defined by

$$d_2(f,g) = \sup_{\xi \in \mathbb{R}^N} \frac{\left|\widehat{f}(\xi) - \widehat{g}(\xi)\right|}{|\xi|^2},$$

will do. We then write

$$d(f(0,\cdot), \mu(t_n,\cdot)) \leq d(f(-t_n), \mu(0,\cdot)).$$

Letting n go to infinity, we get

$$d(f(0,\cdot), m) \leq \underline{\lim}\, d(f(-t_n), \mu(0,\cdot)).$$

But of course, $f(-t_n) \rightharpoonup \mu(0,\cdot)$. Therefore, we can conclude that the right-hand side is 0, *as soon as we know that there is no loss of energy for the sequence* (f^n). This condition means that, at least for some subsequence,

$$\lim_{R\to\infty} \sup_n \int f(-t_n,\cdot)|v|^2 1_{|v|\geq R} = 0. \tag{1}$$

The converse of this condition is exactly that for some $\varepsilon > 0$, for all $R > 0$,

$$\underline{\lim}_{t\to-\infty} \int_{|v|\geq R} f(t,\cdot)|v|^2 \geq \varepsilon,$$

i.e. that a nonnegligible fraction of the energy goes to infinity.

The condition (1) is true for eternal solutions of the Landau equation, as implied by Corollary 6.1 in [7]: more precisely, we show that if f is a solution of the Boltzmann equation and $\chi_f(K) = (1/2)\int f|v|^2 1_{|v|^2/2\geq K}$, then

$$\chi_{f(t,\cdot)}(K) \leq \frac{N}{2}e^{-2t} + \frac{C}{K}. \tag{2}$$

Hence, if f is eternal,

$$\chi_{f(0,\cdot)}(K) \leq \frac{C}{K},$$

and the conclusion follows.

For the Boltzmann equation, this is not so simple, since nothing is known about the uniformity of the decrease of the tails of energy. In fact, we were unable to progress substantially on this problem. Bobylev [1] has proven that for all $\delta > 0$, one can find an initial datum for the Boltzmann equation such that the trend to equilibrium is slower than $C_\delta e^{-\delta t}$. However, examination of the constant C_δ (which is explicit) does not rule out the possibility that some relation like (2) hold with another function $\phi(t)$ instead of e^{-2t}.

Let us now briefly give another strategy, based on a direct use of the distance d_2, which entails the result immediately for the (linear) Fokker-Planck equation.

Proposition 3. *Let f be a solution of the Fokker-Planck equation $\partial_t f = \nabla \cdot (\nabla f + v \cdot f)$. Then,*

$$d_2(f(t,\cdot), M) \leq e^{-2t} d_2(f(0,\cdot), M).$$

Proof. It is immediate : since $f(t,\cdot) = M_{1-e^{-2t}} * f(0,\cdot)_{e^{-2t}}$ (where M_θ is the Maxwellian distribution with temperature θ), we have

$$\widehat{f}(t,\xi) = \widehat{M}\left(\sqrt{1-e^{-2t}}\xi\right)\widehat{f}(0, e^{-t}\xi).$$

Therefore, using $\widehat{M}(\xi) = \widehat{M}(\sqrt{1-e^{-2t}}\xi)\widehat{M}(e^{-t}\xi)$,

$$\begin{aligned} d_2(f(t,\cdot, M)) &= \sup_\xi \frac{\left|\widehat{M}\left(\sqrt{1-e^{-2t}}\xi\right)\widehat{f}(0,e^{-t}\xi) - \widehat{M}(e^{-t}\xi)\right|}{|\xi|^2} \\ &\leq \sup_\xi \frac{\left|\widehat{f}(0,e^{-t}\xi) - \widehat{M}(e^{-t}\xi)\right|}{|\xi|^2} = e^{-2t} d_2(f(0,\cdot), M). \end{aligned}$$

□

Since $d_2(f(0,\cdot), M)$ is bounded by a quantity depending only on the energy of f, we conclude as before that there are no eternal solutions of the Fokker-Planck equation.

Now, if one tries to apply the same method to the Boltzmann equation, writing as in [4]

$$\frac{\partial}{\partial t}\frac{(\widehat{f}-\widehat{M})}{|\xi|^2} + \frac{(\widehat{f}-\widehat{M})}{|\xi|^2} = \int_{S^{N-1}} dn \left[\frac{\widehat{f}(\xi^+)\widehat{f}(\xi^-) - \widehat{M}(\xi)}{|\xi|^2}\right],$$

and using the fact that at least one of the two vectors ξ^+ and ξ^- has norm less than $|\xi|/\sqrt{2}$, one can arrive (at least formally) to the differential inequality

$$\frac{\partial}{\partial t} J(t,R) + J(t,R) \leq J\left(t, \frac{R}{\sqrt{2}}\right), \tag{3}$$

where

$$J(t,R) = \sup_{|\xi|\leq R} \frac{|\widehat{f}(t,\xi) - \widehat{M}(\xi)|}{|\xi|^2}.$$

Therefore, a possible way towards proving conjecture 1 would be to prove that every bounded solution of (3), increasing in R, is in fact identically 0.

Remark. It is easy to check that Bobylev's explicit solutions tend to become negative if one tries to continue them for (too) negative times. This is also true for simple caricatures as the 4-dimensional velocity model. For considerably more complicated simplified models, Cabannes [2] was able to prove Conjecture 1.

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