THE DISCRETE BOLTZMANN EQUATION

Lecture Notes given at the University of California, Berkeley during Spring Quarter 1980

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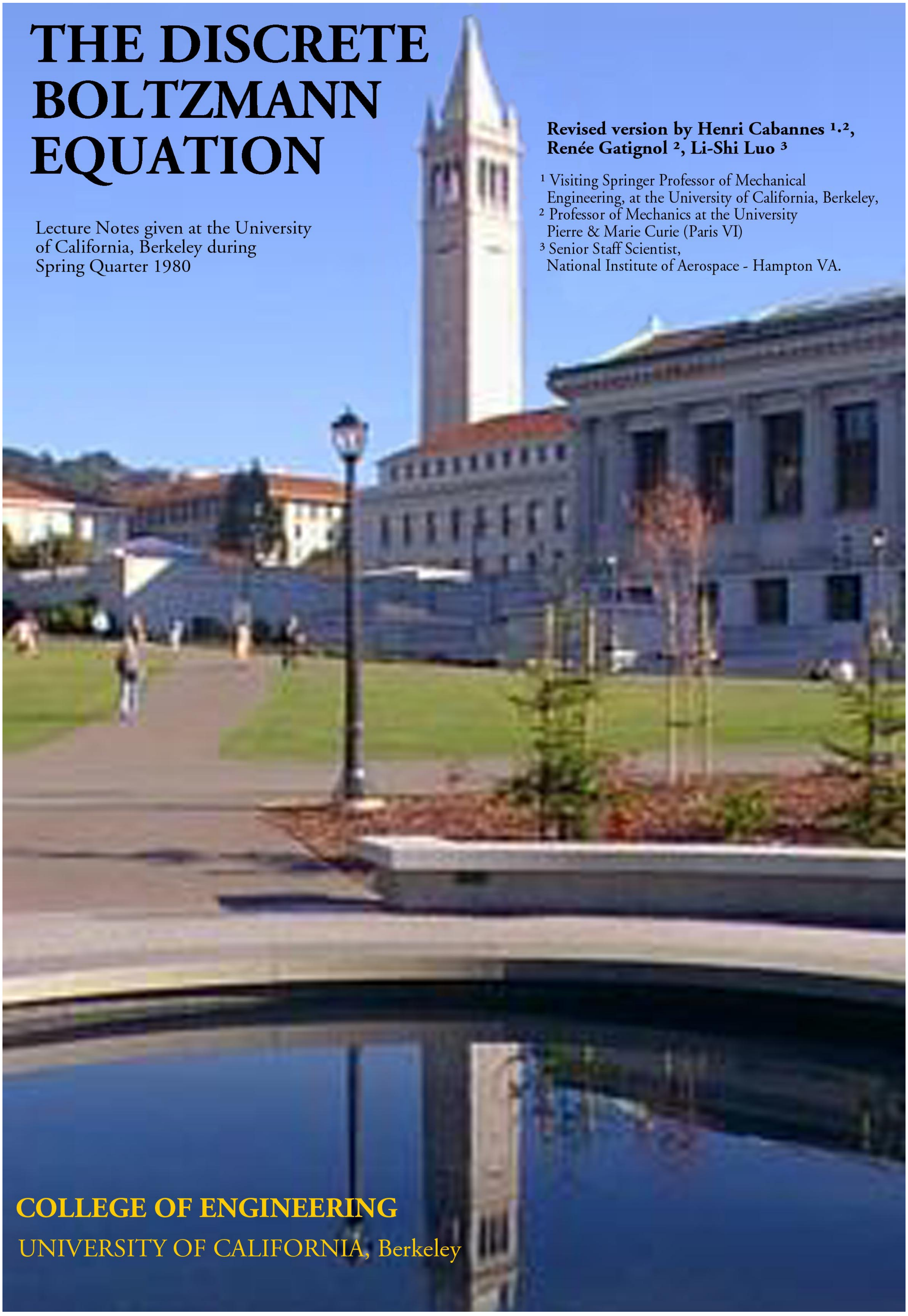

COLLEGE OF ENGINEERING
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The Discrete Boltzmann Equation

(Theory and Applications)

Henri Cabannes*

Lecture Notes given at the University of California at Berkeley
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Contents

LIST OF SYMBOLS AND NOTATIONS

Fonts:

italic:	for scalers;
Bold italic:	for vectors in D-dimensional vector space, $D = 1, 2, 3$;
Boldface:	for vectors in $\mathbb{R}^b$;
Sans serif:	for operators or second rank tensors (matrices);
Bold sans serif:	for matrices with vector/tensor elements;
BLACKBOARD BOLD:	for vector space.

Marks above Symbols:

$\widehat{\ }$	unit vector
$\widetilde{\ }$	multiplied by $e^{\lambda t}$, *e.g.*, $\widetilde{N}_i = e^{\lambda t} N_i$
$\tilde{\ }$	functions periodic in x with period ϖ
$\bar{\ }$	average over $\{\boldsymbol{\xi}_i\}$, *e.g.*, $n\overline{\phi} := \sum_i \phi(\boldsymbol{\xi}_i) f(\boldsymbol{\xi}_i)$

Alphabet Symbols:

A	advection operator, $\mathsf{A}_{ij} = \delta_{ij} \boldsymbol{\xi}_i \cdot \boldsymbol{\nabla}$.
A_{ij}^{kl}	transition probability of a collision $(\boldsymbol{\xi}_i, \boldsymbol{\xi}_j) \to (\boldsymbol{\xi}_k, \boldsymbol{\xi}_l)$
a_{ij}^{kl}	the probability of two particles of $(\boldsymbol{\xi}_i,\ \boldsymbol{\xi}_j)$ given other two particles of $(\boldsymbol{\xi}_k,\ \boldsymbol{\xi}_l)$.
a_i	$a_i := \langle \mathbf{N},\ \mathbf{V}^{(i)} \rangle$
b	the number of discrete velocities
b_j	$b_j := \langle \mathbf{N},\ \mathbf{W}^{(j)} \rangle$
b_{ij}	$N_i = \sum_{j=s+1}^{b} b_{ij} N_j$, $i \leq s$, with the conditions $\sum_{i=1}^{s} N_i V_I^{(k)} = a_k$
$\boldsymbol{c}_i$	the peculiar velocity of the i-th particle, $\boldsymbol{c}_i := (\boldsymbol{\xi}_i - \boldsymbol{u})$.
c_k	$c_k := \langle \ln(\mathbf{N}),\ \mathbf{V}^{(k)} \rangle$
c	basic unit of a discrete velocity in $\{\boldsymbol{\xi}_i\}$
$\mathbb{F}$	the space of summational invariants, $\mathbb{F} = \mathbb{R}^s \subset \mathbb{R}^b$
$\mathbf{F}$	$\mathbf{F}(\mathbf{U},\ \mathbf{V})$, $\mathbf{U},\ \mathbf{V} \in \mathbb{R}^b$, is a bi-linear mapping from $\mathbb{R}^b \times \mathbb{R}^b \to \mathbb{R}^b$
F_i	i-th component of $\mathbf{F} = (F_1,\ F_2,\ \ldots,\ F_b)$.
G_i	the gain of N_i due to collision
$\mathcal{H}$	$\mathcal{H}(N_{s+1},\ \ldots,\ N_b) := H(\sum_{j=s+1}^{b} b_{1j} N_j,\ \ldots,\ N_{s+1},\ \ldots,\ N_b)$

H	H-Boltzmann function, $H := \sum_{i=1}^{b} N_i \ln N_i$
$h(\tau)$	$h := \sum_{i=1}^{b} n_i \ln n_i$
$(i,\, j)$	$(i,\, j) := (\boldsymbol{\xi}_i,\, \boldsymbol{\xi}_j)$
k_B	the Boltzmann constant
$(k,\, l)$	$(k,\, l) := (\boldsymbol{\xi}_k,\, \boldsymbol{\xi}_l)$
L	$\mathsf{L} := \left(\frac{\partial^2 L}{\partial c_k \partial c_l}\right)$
$\boldsymbol{L}_{ij}$	$\boldsymbol{L}_{ij} := \sum_{k=1}^{b} \boldsymbol{\xi}_k V_k^{(i)} V_k^{(j)}$
L_i	the loss of N_i due to collision
M	$\mathsf{M} = \left(\frac{\partial^2 \boldsymbol{M}}{\partial c_k \partial c_l}\right)$
$\boldsymbol{M}_{ij}$	$\boldsymbol{M}_{ij} := \sum_{k=1}^{b} \boldsymbol{\xi}_k W_k^{(i)} V_k^{(j)}$
M_j	the number density of the particles with velocity $\boldsymbol{\zeta}_j$, $M_j(\boldsymbol{x},\, t) := M(\boldsymbol{x},\, \boldsymbol{\zeta}_j,\, t)$
M_{j0}	the initial value $M_{j0}(\boldsymbol{x}) := M_j(\boldsymbol{x},\, 0)$
m	particle mass
$\mathbf{N}$	$\mathbf{N} = (N_1,\, N_2,\, \ldots,\, N_b)$, $\mathbf{N} \in \mathbb{R}^b$.
N_i	the number density of the particles with velocity $\boldsymbol{\xi}_i$, $N_i(\boldsymbol{r},\, t) := N(\boldsymbol{r},\, \boldsymbol{\xi}_i,\, t)$.
$\widetilde{N}_i$	$\widetilde{N}_i(\boldsymbol{x},\, t) := e^{\lambda t} N_i(\boldsymbol{x},\, t)$
n	mean particle number, $n = \sum_i N_i$.
n_i	$n_i := N_i/n$.
P	the pressure tensor
p	the hydrostatic pressure
$\boldsymbol{q}$	the thermal flux
$\mathbb{R}^b$	a b-dimensional real vector space
$\boldsymbol{r}$	position in space, $\boldsymbol{r} = (x,\, y,\, z)$
S	area of a cross section
t	time
$\boldsymbol{u}$	macroscopic (or mean) velocity, $n\boldsymbol{u} = \sum_i N_i \boldsymbol{\xi}_i$.
$\mathbf{V}^{(i)}$	$\mathbf{V}^{(i)} \in \mathbb{F}$, $i = 1,\, 2,\, \ldots,\, s$
$\mathbf{W}^{(j)}$	$\mathbf{W}^{(j)} \notin \mathbb{F}$, $j = s+1,\, s+2,\, \ldots,\, b$

Greek Symbols:

(Δ)	a domain in x-t plane
$\boldsymbol{\zeta}_j$	discrete particle velocity, $\boldsymbol{\zeta}_j = (\zeta_{jx},\, \zeta_{jy},\, \zeta_{jz})$

$\boldsymbol{\xi}$	particle velocity, $\boldsymbol{\xi} = (\xi_x, \xi_y, \xi_z)$
$\boldsymbol{\xi}_i$	discrete particle velocity, $\boldsymbol{\xi}_i = (\xi_{xi}, \xi_{yi}, \xi_{zi})$, $\{\boldsymbol{\xi}_i \vert i = 1, 2, \ldots, b\}$.
ξ^*	wave propagating speed
$(\boldsymbol{\xi}_i, \boldsymbol{\xi}_j)$	a pair of particles with velocities $\boldsymbol{\xi}_i$ and $\boldsymbol{\xi}_j$
ϖ	spatial period of N_i
ρ	mean mass density, $\rho = mn$.
Σ	a surface
τ	$\tau := cSnt$, a dimensionless time
$\boldsymbol{\Phi}$	$\boldsymbol{\Phi} := (\phi_1, \phi_2, \ldots, \phi_b) \in \mathbb{R}^b$
ϕ	$\phi = \phi(\boldsymbol{\xi})$
ϕ_i	$\phi_i = \phi(\boldsymbol{\xi}_i)$

PREFACE TO REVISED EDITION

This set of Lecture Notes on Discrete Boltzmann Equation was used by one of us (H.C.) for a graduate course in the Mechanical Engineering Department of the University of California at Berkeley, in 1980.

The kinetic theory of discrete velocity models (DVM) has advanced significantly since the first edition of the lecture notes was written. New developments have been made in several areas (see, *e.g.*, surveys [46, 61, 58] and collections of papers [37, 4]). First, in the area of mathematical theory of DVM's, various theorems have been proven showing the convergence properties of the DVM's in connection with the Boltzmann equation [7, 6]. Second, along with the growing power of computers, numerical algorithms based on DVM's have been developed to solve kinetic equations and are now used routinely for various applications (*e.g.* [39, 52]). Finally, there have been new developments in using kinetic theory to construct numerical methods for numerical solutions of hydrodynamic equations. The lattice gas cellular automata (LGCA) [43, 33, 32] and lattice Boltzmann equation (LBE) [50, 31, 29, 66] are notable kinetic methods for solving the Navier-Stokes equations. These new methods are closely related to DVM [49] and provide alternatives in the arena of computational fluid dynamics (CFD).

The only definitive monograph on kinetic theory of discrete velocity models was published more than a quarter century ago (in French) [35]. Given the growing interest in DVM's and their connections to kinetic methods such the LBE method, we believe republication of the Lecture Notes is appropriate because of their potential to serve as a primer and a reference for a wider readership.

In addition to correcting typos, several changes have been made in the revised edition of the Lecture Notes: (1) The second edition of the Notes is prepared in LaTeX typesetting; (2) Some notations and symbols have been changed, and a list of symbols is added for convenience; (3) In various references (*e.g.*, [44, 45, 58]), Maxwell has been credited with creating a discrete velocity model. However, after an exhaustive search, we could not find any supporting evidence in Maxwell's work. Therefore, the reference to Maxwell's work on discrete velocity models in the Foreword to the first edition of the Notes is deleted; (4) Sec. 4.8 is included to reflect some recent developments since the Notes were first written in

1980; and (5) References have been updated.

To commemorate an event which bears a historic significance to the authors, we include a photograph of two of us (H.C. and R.G.) and Professor James E. Broadwell of Cal. Tech., taken during the Workshop on Large Nonlinear Systems, held in Santa Fe, New Mexico, on October 27–29, 1986. The Workshop marked the beginning of the LGCA and LBE methods. The proceedings of this workshop have been published in Complex Systems **1**(4), 1987, and remain an important reference in the field.

The Lecture Notes are available to the public in PDF format on our websites.

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From left to Right: Henri Cabannes, James E. Broadwell, and Renée Gatignol, during the Workshop on Large Nonlinear Systems, Santa Fe, New Mexico, October 27–29, 1986.

FOREWORD TO FIRST EDITION

The following lecture notes are developed from a course I gave at the University of California at Berkeley during the Spring Quarter, 1980.

It is a pleasure for me to thank Professor Maurice Holt and Professor Chang-Lin Tien, Chairman of the Mechanical Engineering Department, for inviting me to be a Springer Visiting Professor during this quarter. I also thank all the members of the department for their friendly welcome.

These notes have been written with the very efficient help of Larry Wigton, a graduate student in Applied Mathematics. It has been a great pleasure for me to work with him and I thank him warmly for his help. Thanks are also due to Ms. Loris C.-H. Donahue who typed the manuscript in its present form.

INTRODUCTION

In the study of the kinetic theory of gases, the idea of considering models for which the velocity distribution is discrete is due to Carleman [27]. In 1957, Carleman [27] considered a fictitious gas for which two molecules could interchange their velocities during a collision. Carleman wrote a system of two differential equations for this gas which have several properties similar to those of the Boltzmann equation. In particular, he established an H-theorem.

In 1960, Gross [40] underscored the interest in discrete velocity distributions by showing that they allow one to replace the integral-differential Boltzmann equation by a system of coupled nonlinear partial differential equations. Gross pointed out that this same technique has been successfully applied in the field of radiation transfer.

Simple discrete models are described by Broadwell in 1964 [10, 11]. These models are used to solve problems in the dynamics of rarefied gases for which the Boltzmann equation is applicable. For Couette and Rayleigh flows and for the shock structure at infinite Mach number, these discrete models give a simple physical picture, and the quantitative results are very close to those found by other methods.

In 1965, Gatignol studied shocks that can occur in a gas at rest using one of the models described by Broadwell [10, 11]. Just as in gas dynamics, shocks are accompanied by a compression and an increase in the density, and the velocity of propagation of the shock in a gas at rest is supersonic. For a strong shock, the thickness of the shock is of the order of a few free mean paths. In 1966, Gatignol introduced a regular plane model with $2r$ velocities of the same modulus, and for $r = 4$ she studied the shocks which appear in a gas at rest and demonstrated the existence of expansion waves. The relevant references are given in the monograph by Gatignol [35].

In 1966, Harris [44] considered a moderately dense gas described by a regular plane model with six velocities. An H-theorem taking into account ternary collisions is proved, leading to the hope of a similar theorem in classical kinetic theory. Actually, Harris proved an H-theorem for a gas in which only binary collisions and a part of the ternary collisions can occur. This result was generalized by Gatignol in 1969 [34] to cover the case in which both binary and all the ternary collisions can occur. In section 3.1.3, we give a very simple proof, for a regular plane model with four velocities and only binary collisions, that the successive derivatives of the H-Boltzmann function alternate in sign, which was first attempted but failed by Harris [45]. We hope this result can be extended to more general models.

In 1971, Godunov and Sultangazin [38] considered the Broadwell model with six velocities [10]. They pointed out the relation between the kinetic equations derived from this model and the original Boltzmann equation. Furthermore, they studied a system of modified kinetic equations and obtained an existence theorem for the solutions of this system. These solutions satisfy in a certain sense the appropriate Euler equations. In 1975, Gatignol obtained similar results for the general model with a discrete velocity distribution [36].

From the point of view of statistical mechanics, in 1972, Hardy and Pomeau [43] derived the Euler and Navier-Stokes equations starting with the Liouville equation and using a regular plane model with four velocities, which is the first lattice-gas cellular automaton (LGCA) model we know of.

In the kinetic theory of gases, the method of discrete velocities has been discussed by a number of authors, *e.g.*, Kogan [47], Guiraud [41, 42] and Smolderen [60]. The first systematic treatment of the subject is the monograph written by Gatignol, *Théorie Cinétique des Gaz à Répartition Discrète de Vitesses*, published as the *Lecture Notes in Physics*, Vol. **36**, by Springer in 1975 [35].

In her monograph [35] of four chapters, Gatignol begins with the presentation of the general model of a gas with discrete velocities. Only binary collisions are considered in general, and the original Boltzmann equation is replaced by a system of coupled nonlinear partial differential equations. This system is shown to have interesting properties, some of which are similar to those of the Boltzmann equation. An appropriate definition of the summational invariants allows her to introduce independent macroscopic state variables analogous to the density, velocity and temperature of a classical gas (however there may be more than $(D+2)$ summational invariants in D-dimensional space). She then writes conservation equations for the macroscopic state variables which in general form a non closed system of equations. A suitably defined H-Boltzmann function leads to an H-theorem and to the notion of Maxwellian state. For a gas in a particular microscopic state there exists one and only one Maxwellian state having the same macroscopic variables. Some models are then described and the proof by Harris concerning the successive derivatives of the H-Boltzmann function is described [45]. (Unfortunately, the proof in [45] is erroneous.)

All the results discussed above by Gatignol [35] also appear in these Notes. In fact Chapters 1 to 3 are devoted to these topics. Chapter 4 of the Notes discusses recent developments which do not appear in Gatignol's monograph [35].

In Chapters 2 to 4 of her monograph [35], Gatignol studies the Euler equations for the general case, and the Navier-Stokes equations for some particular models. The Euler equations are written in a symmetric form which demonstrates their hyperbolic character. The propagation of weak shocks is studied from two points of view: First in a medium without dissipation by using the Euler equations and the Lax criterion [48], and second in a medium with dissipation by applying the necessary conditions for existence of a shock structure. The compatibility of the results of these two methods is demonstrated.

In the last Chapter of the Notes (Chapter 4), we study the initial value problem. This problem has of course a local solution. In general, when we have a local solution we do not know if it exists globally (that is for all time). It is generally impossible to prove the existence of a global solution except for very simple equations. It is remarkable that global existence can be proved for the discrete Boltzmann equation.

Chapter 1

PRESENTATION OF THE GENERAL MODEL

In this Chapter the general model of a gas with a discrete distribution of velocities is presented. The medium is composed of identical particles which can only have velocity vectors belonging to a finite set of b vectors. When the medium is sufficiently rarefied, only binary collisions are considered. The original Boltzmann equation is replaced by a system of b first order partial differential equations. Each equation in the system is linear with respect to the derivatives of the unknown functions and quadratic with respect to the functions themselves. Some properties which are essential for the Boltzmann equation are established for this system.

1.1 Evolution Equations of Discrete Gases

1.1.1 Binary Collisions

We establish the usual Cartesian coordinate system $(x,\ y,\ z)$ in space. The position will be denoted by $\boldsymbol{r}$ and the time by t. The medium is composed of identical particles, each with mass m and a velocity belonging to the set of b vectors:

$$\{\boldsymbol{\xi}_i | i = 1,\ 2, \ldots, b\}.$$

Two particles are said to be in interaction or in collision when the distance separating them is smaller than a certain length representing the radius of action of the intermolecular potential. We are only interested in the velocities of the particles before the collision: $\boldsymbol{\xi}_i$ and $\boldsymbol{\xi}_j$, and after the collision: $\boldsymbol{\xi}_k$ and $\boldsymbol{\xi}_l$. These four velocities are not arbitrary because during a collision, the momentum and energy must be conserved:

$$\left.\begin{aligned} \boldsymbol{\xi}_i + \boldsymbol{\xi}_j &= \boldsymbol{\xi}_k + \boldsymbol{\xi}_l, \\ \boldsymbol{\xi}_i^2 + \boldsymbol{\xi}_j^2 &= \boldsymbol{\xi}_k^2 + \boldsymbol{\xi}_l^2. \end{aligned}\right\} \tag{1.1.1-1}$$

Such a collision will be denoted as $(\boldsymbol{\xi}_i, \boldsymbol{\xi}_j) \to (\boldsymbol{\xi}_k, \boldsymbol{\xi}_l)$, or simply $(i,\, j) \to (k,\, l)$. Given the couple $(\boldsymbol{\xi}_i, \boldsymbol{\xi}_j)$, we must look for the velocities $(\boldsymbol{\xi}_k, \boldsymbol{\xi}_l)$ which satisfy (1.1.1-1) and belong to the given set of b vectors. This problem has to be solved in each particular case. The similar problem of classical kinetic theory admits an infinite family of solutions depending on two parameters.

From (1.1.1-1) we deduce:

$$\begin{gathered} (\boldsymbol{\xi}_i + \boldsymbol{\xi}_j)^2 = (\boldsymbol{\xi}_k + \boldsymbol{\xi}_l)^2, \\ \boldsymbol{\xi}_i \cdot \boldsymbol{\xi}_j = \boldsymbol{\xi}_k \cdot \boldsymbol{\xi}_l, \\ (\boldsymbol{\xi}_i - \boldsymbol{\xi}_j)^2 = (\boldsymbol{\xi}_k - \boldsymbol{\xi}_l)^2. \end{gathered}$$

Consequently the sphere with diameter $\|\boldsymbol{\xi}_i - \boldsymbol{\xi}_j\|$ is identical to that with diameter $\|\boldsymbol{\xi}_k - \boldsymbol{\xi}_l\|$.

The numerical density of the particles with velocity $\boldsymbol{\xi}_i$ will be denoted as $N_i(\boldsymbol{r},\, t)$, $\boldsymbol{r}$ being the position and t the time. When the collision time is very short with respect to the mean free time of flight (and is also short compared to the macroscopic time) and when the radius of action of the inter-molecular potential is very small with respect to the mean free path (and is also small compared to the macroscopic reference length), the number of collisions during the time interval $(t,\, t + dt)$ and in the volume $d\boldsymbol{r}$ surrounding the point $\boldsymbol{r}$ between particles with velocities $\boldsymbol{\xi}_i$ and $\boldsymbol{\xi}_j$ before collision, and yielding particles with velocities $\boldsymbol{\xi}_k$ and $\boldsymbol{\xi}_l$ after collision is proportional to $N_i N_j$ and in fact equal to:

$$A_{ij}^{kl} N_i N_j \, dt \, d\boldsymbol{r},$$

where the coefficient A_{ij}^{kl} is called the transition probability of the collision $(\boldsymbol{\xi}_i, \boldsymbol{\xi}_j) \to (\boldsymbol{\xi}_k, \boldsymbol{\xi}_l)$.

When the particles are point masses with a finite radius of action, the particles with velocity $\boldsymbol{\xi}_j$ are able to meet a particle with velocity $\boldsymbol{\xi}_i$ during the time dt which is in a cylinder of cross sectional area denoted by S and of height $\|\boldsymbol{\xi}_i - \boldsymbol{\xi}_j\| dt$. If we assume that the couple of particles with velocities $\boldsymbol{\xi}_i$ and $\boldsymbol{\xi}_j$ gives the couple of particles with velocities $\boldsymbol{\xi}_k$ and $\boldsymbol{\xi}_l$ with probability

$$A_{ij}^{kl} N_i N_j \, dt \, d\boldsymbol{r} = S \|\boldsymbol{\xi}_i - \boldsymbol{\xi}_j\| dt N_j a_{ij}^{kl} N_i \, d\boldsymbol{r},$$

such that

$$A_{ij}^{kl} = S|\boldsymbol{\xi}_i - \boldsymbol{\xi}_j| a_{ij}^{kl}, \qquad \text{with} \qquad \sum_{(k,l)} a_{ij}^{kl} = 1, \tag{1.1.1-2}$$

where the summation is only performed over possible couple $(k,\, l)$. In practice we assume that, in a collision between particles with velocities $\boldsymbol{\xi}_i$ and $\boldsymbol{\xi}_j$, all possible couples after collision are obtained with the same probability. Therefore it is enough to know the number of couples which can be obtained after the collision to compute the coefficients a_{ij}^{kl} and A_{ij}^{kl}.

The transition probabilities A_{ij}^{kl} are non-negative constants (the value zero being given to all non-realizable collisions), symmetric with respect to the indices i and j, and k and l:

$$A_{ij}^{kl} = A_{ji}^{kl} = A_{ij}^{lk}. \tag{1.1.1-3}$$

As a consequence of the discrete model, we also have $a_{ij}^{kl} = a_{kl}^{ij}$ which gives:

$$A_{ij}^{kl} = A_{kl}^{ij}. \tag{1.1.1-4}$$

This means that the collision $(\boldsymbol{\xi}_i, \boldsymbol{\xi}_j) \to (\boldsymbol{\xi}_k, \boldsymbol{\xi}_l)$ and the inverse collision $(\boldsymbol{\xi}_k, \boldsymbol{\xi}_l) \to (\boldsymbol{\xi}_i, \boldsymbol{\xi}_j)$ occur with the same probability. Relation (1.1.1-4) is called the relation of micro-reversibility. It is possible to introduce hypothesis less restricted than the micro-reversibility.

1.1.2 Kinetic Equations

In order to derive the equations which describe the evolution of the gas in the absence of external forces, we make the hypothesis that there is no correlation between colliding particles before they enter the interaction domain (hypothesis of molecular chaos).

The balance equation for particles with velocity $\boldsymbol{\xi}_i$ (with density function N_i) is:

$$\partial_t N_i + \boldsymbol{\xi}_i \cdot \boldsymbol{\nabla} N_i = G_i - L_i,$$

where L_i and G_i represent respectively the lose and gain of molecules with velocity $\boldsymbol{\xi}_i$ due to collisions during the unit time and in the unit volume at the time t and at the point $\boldsymbol{r}$. A molecule with velocity $\boldsymbol{\xi}_i$ is lost when this molecule encounters another molecule with velocity $\boldsymbol{\xi}_j$ $(j \neq i)$ and the resulting collision is nontrivial [that means $(\boldsymbol{\xi}_i, \boldsymbol{\xi}_j) \to (\boldsymbol{\xi}_k, \boldsymbol{\xi}_l)$ where the couple $(\boldsymbol{\xi}_k, \boldsymbol{\xi}_l)$ is different from the couple $(\boldsymbol{\xi}_i, \boldsymbol{\xi}_j)$]. Conversely, there is a gain of a particle with velocity $\boldsymbol{\xi}_i$ when one is produced by a nontrivial collision. Thus we obtain:

$$G_i - L_i = \sum_{j=1}^{b} \sum_{(k,l)} \left(A_{kl}^{ij} N_k N_l - A_{ij}^{kl} N_i N_j \right).$$

If we use the relation of micro-reversibility (1.1.1-4) and if we perform the summation over the indices j, k, and l instead of over the couples $(k,\, l)$, we can rewrite the kinetic equations in the following form:

$$\partial_t N_i + \boldsymbol{\xi}_i \cdot \boldsymbol{\nabla} N_i = \frac{1}{2} \sum_{j,k,l} A_{ij}^{kl} (N_k N_l - N_i N_j), \qquad \text{for } i = 1,\, 2,\, \ldots,\, b. \tag{1.1.2-1}$$

The system of kinetic equations can be written in a compact form by introducing a symmetric bilinear operator $\mathbf{F}$. The mapping

$$(\mathbf{U},\, \mathbf{V}) \to \mathbf{F}(\mathbf{U},\, \mathbf{V})$$

of $\mathbb{R}^b \times \mathbb{R}^b$ into $\mathbb{R}^b$ is defined by:

$$F_i(\mathbf{U},\, \mathbf{V}) = \frac{1}{4} \sum_{j,k,l} A_{ij}^{kl} \left(U_k V_l + U_l V_k - U_i V_j - U_j V_i \right), \tag{1.1.2-2}$$

where F_i is the i-th component of the vector $\mathbf{F}$ in $\mathbb{R}^b$.

We denote by $\mathbf{N}$ the vector in $\mathbb{R}^b$ with components $\{N_i\}$, and by A the diagonal matrix of order b with $\mathsf{A}_{ij} = \delta_{ij} \boldsymbol{\xi}_i \cdot \boldsymbol{\nabla}$. We can rewrite Eq. (1.1.2-1) as the following:

$$\partial_t \mathbf{N} + \mathsf{A} \cdot \mathbf{N} = \mathbf{F}(\mathbf{N},\, \mathbf{N}). \tag{1.1.2-3}$$

This equation plays the role of the Boltzmann equation and is called the discrete Boltzmann equation. The evolution of a gas with a discrete distribution of velocities is described by a system of coupled semi-linear partial differential equations.

1.1.3 Properties of the Collision Operator

Given two vectors in $\mathbb{R}^b$, namely $\mathbf{X} = (X_1, X_2, \ldots, X_b)$ and $\mathbf{Y} = (Y_1, Y_2, \ldots, Y_b)$, we define their scalar product by

$$\langle \mathbf{X}, \mathbf{Y} \rangle = X_1 Y_1 + X_2 Y_2 + \ldots + X_b Y_b.$$

Given a vector $\mathbf{\Phi} := (\phi_1, \phi_2, \ldots, \phi_b)$, the scalar product $\langle \mathbf{\Phi}, \mathbf{F}(\mathbf{U}, \mathbf{V}) \rangle$ is equal to any one of the following expressions:

$$\left. \begin{aligned} &\frac{1}{4} \sum_{i,j,k,l} \phi_i \left[A_{kl}^{ij}(U_k V_l + U_l V_k) - A_{ij}^{kl}(U_i V_j + U_j V_i) \right], \\ &\frac{1}{4} \sum_{i,j,k,l} \phi_j \left[A_{kl}^{ij}(U_k V_l + U_l V_k) - A_{ij}^{kl}(U_i V_j + U_j V_i) \right], \\ -&\frac{1}{4} \sum_{i,j,k,l} \phi_k \left[A_{kl}^{ij}(U_k V_l + U_l V_k) - A_{ij}^{kl}(U_i V_j + U_j V_i) \right], \\ -&\frac{1}{4} \sum_{i,j,k,l} \phi_l \left[A_{kl}^{ij}(U_k V_l + U_l V_k) - A_{ij}^{kl}(U_i V_j + U_j V_i) \right]. \end{aligned} \right\} \tag{1.1.3-1}$$

The first formula follows from the definition of the scalar product (and relation (1.1.1-4)), the second is deduced from the first by exchanging indices i and j, the third is deduced from the first by exchanging the couples $(i,\, j)$ and $(k,\, l)$, and finally the last formula is deduced from the third by exchanging indices k and l. Adding all the equations in (1.1.3-1) yields:

$$\langle \mathbf{\Phi}, \mathbf{F}(\mathbf{U}, \mathbf{V}) \rangle = \frac{1}{16} \sum_{i,j,k,l} (\phi_i + \phi_j - \phi_k - \phi_l) \left[A_{kl}^{ij}(U_k V_l + U_l V_k) - A_{ij}^{kl}(U_i V_j + U_j V_i) \right]. \tag{1.1.3-2}$$

1.2 The Macroscopic Variables

1.2.1 The Macroscopic Variables

The microscopic description of the gas is given by the knowledge of all the densities N_i. However its macroscopic behavior depends only on quantities called mean values. Because the velocities have only b possible values, any function ϕ of the velocity can only take on the b values $\phi_i = \phi(\boldsymbol{\xi}_i)$, $i = 1,\, 2,\, \ldots,\, b$. In this way the function ϕ of velocity is associated with a vector in $\mathbb{R}^b$. We define the mean value $\bar{\phi}$ by:

$$n\bar{\phi} = \sum_{i=1}^{b} N_i \phi_i = \langle \mathbf{N},\, \mathbf{\Phi} \rangle, \qquad \text{with} \qquad n = \sum_{i=1}^{b} N_i. \tag{1.2.1-1}$$

The mean $\bar{\phi}$ has the same tensorial nature as each component ϕ_i of $\mathbf{\Phi}$, with $\mathbf{\Phi} := (\phi_1,\, \phi_2,\, \ldots,\, \phi_b)$.

The most interesting macroscopic variables are the mean velocity $\boldsymbol{u}$ of the gas, the pressure tensor P, the hydrostatic pressure p, the temperature T, and the thermal flux $\boldsymbol{q}$:

$$\left.\begin{aligned}
\boldsymbol{u} &= \frac{1}{n}\sum_{i=1}^{b} N_i\boldsymbol{\xi}_i, \\
\mathsf{P} &= m\sum_{i=1}^{b} N_i(\boldsymbol{\xi}_i-\boldsymbol{u})\otimes(\boldsymbol{\xi}_i-\boldsymbol{u}) = m\sum_{i=1}^{b} N_i\boldsymbol{c}_i\otimes\boldsymbol{c}_i, \\
p &= \frac{m}{3}\sum_{i=1}^{b} N_i(\boldsymbol{\xi}_i-\boldsymbol{u})^2 = \frac{m}{3}\sum_{i=1}^{b} N_i\boldsymbol{c}_i^2 = nk_BT, \\
\boldsymbol{q} &= \frac{m}{2}\sum_{i=1}^{b} N_i(\boldsymbol{\xi}_i-\boldsymbol{u})^2(\boldsymbol{\xi}_i-\boldsymbol{u}) = \frac{m}{2}\sum_{i=1}^{b} N_i\boldsymbol{c}_i^2\boldsymbol{c}_i,
\end{aligned}\right\} \tag{1.2.1-2}$$

where $\boldsymbol{c}_i \equiv (\boldsymbol{\xi}_i - \boldsymbol{u})$ is the peculiar velocity of the i-th particle. The temperature can also be defined as:

$$T = \frac{m}{3k_B}\left[\frac{1}{n}\left(\sum_{i=1}^{b} N_i\boldsymbol{\xi}_i^2\right) - \boldsymbol{u}^2\right], \tag{1.2.1-3}$$

where m is the mass of the particles and k_B is the Boltzmann constant. When the velocities have the same modulus $|\boldsymbol{\xi}_i| = c$, we have:

$$T = \frac{m}{3k_B}(c^2 - \boldsymbol{u}^2) \leq \frac{mc^2}{3k_B} = T_M. \tag{1.2.1-4}$$

Thus the temperature is a function of the mean velocity $\boldsymbol{u}$ and the maximum value of the temperature T_M corresponds to a gas at rest.

It should be remarked that the temperature T defined in Eq. (1.2.1-3) is called kinetic temperature and it differs from the thermodynamic temperature. As pointed out by Cercignani [28], the definition of the temperature given in continuous kinetic theory is no longer valid in discrete kinetic theory.

1.2.2 Transport Equations

The equation obtained by multiplying both sides of Eq. (1.1.2-1) by ϕ_i and summing over the index i is called the transport equation for the macroscopic variable ϕ. This is the same as the equation obtained through scalar multiplication of both sides of Eq. (1.1.2-3) by $\boldsymbol{\Phi}$:

$$\langle\boldsymbol{\Phi}, \partial_t\mathbf{N}\rangle + \langle\boldsymbol{\Phi}, \mathsf{A}\cdot\mathbf{N}\rangle = \langle\boldsymbol{\Phi}, \mathbf{F}(\mathbf{N},\mathbf{N})\rangle. \tag{1.2.2-1}$$

The right hand side can be evaluated by formula (1.1.3-2) and the left hand side can be expressed in terms of the mean value $\bar{\phi}$. Indeed we have:

$$\begin{aligned}\langle \mathbf{\Phi}, \partial_t \mathbf{N}\rangle &= \partial_t\langle \mathbf{\Phi}, \mathbf{N}\rangle - \langle \partial_t\mathbf{\Phi}, \mathbf{N}\rangle = \partial_t(n\bar{\phi}) - n\partial_t\bar{\phi},\\ \langle \mathbf{\Phi}, \mathsf{A}\cdot\mathbf{N}\rangle &= \sum_{i=1}^{b}\phi_i\boldsymbol{\xi}_i\cdot\boldsymbol{\nabla}N_i = \boldsymbol{\nabla}\cdot(n\overline{\phi\boldsymbol{\xi}}) - n\overline{\boldsymbol{\xi}\cdot\boldsymbol{\nabla}\phi}.\end{aligned}$$

In the above equations $\overline{\phi\boldsymbol{\xi}}$ denotes the mean value of the vector in $\mathbb{R}^b$ with components $\phi_i\boldsymbol{\xi}_i$, and $\overline{\boldsymbol{\xi}\cdot\boldsymbol{\nabla}\phi}$ is the mean value of the vector with components $\boldsymbol{\xi}_i\cdot\boldsymbol{\nabla}\phi_i$. The transport equation (1.2.2-1) can now be written in the form:

$$\partial_t(n\bar{\phi}) + \boldsymbol{\nabla}\cdot(n\overline{\phi\boldsymbol{\xi}}) - n(\overline{\partial_t\phi} + \overline{\boldsymbol{\xi}\cdot\boldsymbol{\nabla}\phi}) = \frac{1}{8}\sum_{j,k,l}(\phi_i+\phi_j-\phi_k-\phi_l)A_{ij}^{kl}(N_kN_l - N_iN_j). \quad (1.2.2\text{-}2)$$

1.3 The Summational Invariants

1.3.1 Summational Invariants

A summational invariant is a functional of velocity which remains constant during a collision. As the velocities can take on only b distinct values, the summational invariants are the elements of $\mathbb{R}^b$ for which:

$$A_{ij}^{kl}(\phi_i+\phi_j-\phi_k-\phi_l) = 0, \qquad \forall\ i,\ j,\ k,\ l. \quad (1.3.1\text{-}1)$$

The solutions of this system form a vector space $\mathbb{F}$ called the space of summational invariants. The dimension s of $\mathbb{F}$ is at least 1 because the vector with equal components is a solution of Eq. (1.3.1-1). Also s is at most equal to b because $\mathbb{F}$ is a subspace of $\mathbb{R}^b$. The following elements of $\mathbb{R}^b$, which are formed by the x, y, and z components of $\{\boldsymbol{\xi}_i\}$,

$$(\xi_{x1},\,\xi_{x2},\,\ldots,\,\xi_{xb}), \qquad (\xi_{y1},\,\xi_{y2},\,\ldots,\,\xi_{yb}), \qquad (\xi_{z1},\,\xi_{z2},\,\ldots,\,\xi_{zb}),$$

are summational invariants, and so is the element of $\mathbb{R}^b$ with components $\frac{1}{2}m\boldsymbol{\xi}_i^2$.

It is convenient to introduce orthonormal bases for the vector spaces $\mathbb{F}$ and $\mathbb{R}^b$:

$$\begin{aligned}&\text{basis of } \mathbb{F}: && \mathbf{V}^{(1)},\ \mathbf{V}^{(2)},\ \ldots,\ \mathbf{V}^{(s)},\\ &\text{basis of } \mathbb{R}^b: && \mathbf{V}^{(1)},\ \mathbf{V}^{(2)},\ \ldots,\ \mathbf{V}^{(s)},\ \mathbf{W}^{(s+1)},\ \ldots,\ \mathbf{W}^{(b)},\end{aligned}$$

where $\mathbf{V}^{(1)}$ is always taken to be the vector with all components equal to $1/\sqrt{b}$.

The vector $\mathbf{N}$ can be expressed as:

$$\mathbf{N} = \sum_{i=1}^{s} a_i \mathbf{V}^{(i)} + \sum_{j=s+1}^{b} b_j \mathbf{W}^{(j)}, \tag{1.3.1-2}$$

where we have, of course:

$$\left.\begin{array}{ll} a_i = \langle \mathbf{N}, \mathbf{V}^{(i)} \rangle, & i = 1,\, 2,\, \ldots,\, s, \\ b_j = \langle \mathbf{N}, \mathbf{W}^{(j)} \rangle, & j = s+1,\, s+2,\, \ldots,\, b. \end{array}\right\} \tag{1.3.1-3}$$

Theorem 1.1 The following three statements are equivalent:

(a) $\mathbf{\Phi} \in \mathbb{F}$;

(b) $\langle \mathbf{\Phi}, \mathbf{F}(\mathbf{U}, \mathbf{V}) \rangle = 0$, $\forall\ \mathbf{U},\ \mathbf{V} \in \mathbb{R}^b$;

(c) $\langle \mathbf{\Phi}, \mathbf{F}(\mathbf{N}, \mathbf{N}) \rangle = 0$, $\forall\, \mathbf{N} \in \mathbb{R}^b$.

Obviously (a) $\rightarrow$ (b) $\rightarrow$ (c). To complete the proof of the theorem we will show that (c) $\rightarrow$ (a). Relation (1.1.3-2) can be written as:

$$\langle \mathbf{\Phi}, \mathbf{F}(\mathbf{N}, \mathbf{N}) \rangle = \frac{1}{8} \sum_{i,j,k,l} (\phi_i + \phi_j - \phi_k - \phi_l) A_{ij}^{kl} (N_k N_l - N_i N_j).$$

But:

$$\sum_{i,j,k,l} (\phi_i + \phi_j - \phi_k - \phi_l) A_{ij}^{kl} N_k N_l = \sum_{i,j,k,l} (\phi_k + \phi_l - \phi_i - \phi_j) A_{ij}^{kl} N_i N_j,$$

so we have:

$$\begin{aligned} \langle \mathbf{\Phi}, \mathbf{F}(\mathbf{N}, \mathbf{N}) \rangle &= \frac{1}{4} \sum_{i,j,k,l} (\phi_i + \phi_j - \phi_k - \phi_l) A_{ij}^{kl} N_k N_l \\ &= \sum_{(k,\,l)} \sum_{(i,\,j)} (\phi_i + \phi_j - \phi_k - \phi_l) A_{ij}^{kl} N_k N_l = 0. \end{aligned} \tag{1.3.1-4}$$

The last expression is a homogeneous polynomial of second degree with respect to the $\{N_i\}$. This polynomial is identically zero if and only if:

$$\sum_{(i,\,j)} (\phi_i + \phi_j - \phi_k - \phi_l) A_{ij}^{kl} = 0, \tag{1.3.1-5}$$

from which we deduce by multiplying by $(\phi_k + \phi_l)$ and summing on all the couples $(k,\, l)$:

$$\sum_{(i,\,j)} \sum_{(k,\,l)} A_{ij}^{kl} (\phi_k + \phi_l)(\phi_i + \phi_j - \phi_k - \phi_l) = 0, \tag{1.3.1-6}$$

or, by inverting the couples $(i,\, j)$ and $(k,\, l)$:

$$\sum_{(i,j)} \sum_{(k,l)} A_{ij}^{kl} (\phi_i + \phi_j)(\phi_i + \phi_j - \phi_k - \phi_l) = 0, \tag{1.3.1-7}$$

and finally (by taking the difference between the last two expressions):

$$\sum_{(i,j)} \sum_{(k,l)} A_{ij}^{kl} (\phi_i + \phi_j - \phi_k - \phi_l)^2 = 0. \tag{1.3.1-8}$$

The solutions of (1.3.1-8) are precisely the summational invariants.

1.3.2 Conservation Equations

When $\mathbf{\Phi}$ is an element of $\mathbb{R}^b$ independent of time and position, the associated transport equation (1.2.2-1) is a conservation equation if the right hand side is zero. From our previous results this occurs if and only if $\mathbf{\Phi}$ is a summational invariant. In this case we have:

$$\partial_t \langle \mathbf{\Phi}, \mathbf{N} \rangle + \langle \mathbf{\Phi}, \mathsf{A} \cdot \mathbf{N} \rangle = 0, \qquad \forall\, \mathbf{\Phi} \in \mathbb{F}, \tag{1.3.2-1}$$

where we have used the fact that $\mathbf{\Phi}$ is independent of time.

The number of conservation equations is equal to the dimension s of the space $\mathbb{F}$. To each summational invariant corresponds a conservation equation. If we replace $\mathbf{\Phi}$ by $\mathbf{V}^{(k)}$, equation (1.3.2-1) becomes:

$$\partial_t a_k + \sum_{i=1}^{s} \boldsymbol{L}_{ki} \cdot \boldsymbol{\nabla} a_i + \sum_{j=s+1}^{b} \boldsymbol{M}_{kj} \cdot \boldsymbol{\nabla} b_j = 0, \tag{1.3.2-2}$$

where $\boldsymbol{L}_{ki}$ and $\boldsymbol{M}_{kj}$ are the constant vectors:

$$\boldsymbol{L}_{ki} = \sum_{n=1}^{b} \boldsymbol{\xi}_n V_n^{(i)} V_n^{(k)}, \qquad \boldsymbol{M}_{kj} = \sum_{n=1}^{b} \boldsymbol{\xi}_n W_n^{(j)} V_n^{(k)}. \tag{1.3.2-3}$$

The s conservation equations contains b unknown functions, namely, the s functions $\{a_i\}$ and the $(b-s)$ functions $\{b_j\}$. This system of equations is not closed in general because b is usually larger than s. The time derivative ∂_t operates only on the a_i while the space derivatives operate on both the a_i and b_j. Knowledge of the a_i and b_j, which means knowledge of the vector $\mathbf{N}$ (or of the densities $\{N_i\}$) corresponds to the microscopic description of the gas. The quantities a_i are called macroscopic state variables of the gas. In the classical kinetic theory the macroscopic state variables are the density $\rho = mn$, the velocity $\boldsymbol{u}$ and the temperature T.

From equation (1.2.2-2) we can derive the classical conservation laws. Indeed if in equation (1.2.2-2) we take $\boldsymbol{\Phi}$ to be the vector with all components equal to m, we find:

$$\partial_t \rho + \boldsymbol{\nabla}\cdot(\rho \boldsymbol{u}) = 0. \tag{1.3.2-4}$$

If we next choose $\boldsymbol{\Phi}$ to be the vector with components $m\xi_{xi}$ (note ξ_{xi} is the projection of $\boldsymbol{\xi}_i$ on x-axis), we have:

$$\begin{aligned}
&n\bar{\phi} = \rho \boldsymbol{u}, \\
&n\overline{\phi \boldsymbol{\xi}} = m \sum_{i=1}^{b} N_i \xi_{xi} \boldsymbol{\xi}_i, \\
&\overline{\partial_t \phi} = 0 \qquad \text{and} \qquad \overline{\boldsymbol{\xi}\cdot\boldsymbol{\nabla}\phi} = 0.
\end{aligned}$$

Therefore we have:

$$\boldsymbol{\nabla}\cdot(n\overline{\phi \boldsymbol{\xi}}) = m \sum_{i=1}^{b} \xi_{xi} \boldsymbol{\xi}_i \cdot \boldsymbol{\nabla} N_i.$$

On the other hand:

$$\begin{aligned}
\rho \boldsymbol{u} \otimes \boldsymbol{u} + \mathsf{P} &= \rho \boldsymbol{u} \otimes \boldsymbol{u} + m \sum_{i=1}^{b} N_i (\boldsymbol{\xi}_i - \boldsymbol{u}) \otimes (\boldsymbol{\xi}_i - \boldsymbol{u}) \\
&= m \sum_{i=1}^{b} N_i \boldsymbol{\xi}_i \otimes \boldsymbol{\xi}_i.
\end{aligned}$$

The projection of the vector $\boldsymbol{\nabla}\cdot(\rho \boldsymbol{u} \otimes \boldsymbol{u} + \mathsf{P})$ on x-axis is

$$m \sum_{i=1}^{b} \xi_{xi} \boldsymbol{\xi}_i \cdot \boldsymbol{\nabla} N_i.$$

It follows that if we choose $\boldsymbol{\Phi}$ to be the vector with components $m\xi_{i\alpha}$, $\alpha \in \{x, y, z\}$, in (1.2.2-2) we will find:

$$\partial_t (\rho \boldsymbol{u}) + \boldsymbol{\nabla}\cdot(\rho \boldsymbol{u} \otimes \boldsymbol{u} + \mathsf{P}) = 0. \tag{1.3.2-5}$$

Finally, by choosing $\mathbf{\Phi}$ to be the vector in $\mathbb{R}^b$ with components $\frac{1}{2}m\boldsymbol{\xi}_i^2$ we have:

$$\begin{aligned}
n\bar{\phi} &= \sum_{i=1}^{b} N_i \frac{1}{2}m(\boldsymbol{\xi}_i - \boldsymbol{u} + \boldsymbol{u})^2 = \frac{1}{2}m \sum_{i=1}^{b} N_i(\boldsymbol{c}_i + \boldsymbol{u})^2 \\
&= \frac{1}{2}m \sum_{i=1}^{b} N_i(\boldsymbol{c}_i^2 + 2\boldsymbol{c}_i\cdot\boldsymbol{u} + \boldsymbol{u}^2) \\
&= \frac{3}{2}p + \frac{1}{2}\rho\boldsymbol{u}^2, \\
n\overline{\phi\boldsymbol{\xi}} &= \sum_{i=1}^{b} N_i \frac{1}{2}m(\boldsymbol{\xi}_i - \boldsymbol{u} + \boldsymbol{u})^2(\boldsymbol{\xi}_i - \boldsymbol{u} + \boldsymbol{u}) = \frac{1}{2}m \sum_{i=1}^{b} N_i(\boldsymbol{c}_i + \boldsymbol{u})^2(\boldsymbol{c}_i + \boldsymbol{u}) \\
&= \frac{1}{2}m \sum_{i=1}^{b} N_i \left(\boldsymbol{c}_i^2\boldsymbol{c}_i + \boldsymbol{c}_i^2\boldsymbol{u} + \boldsymbol{u}^2\boldsymbol{\xi}_i + 2\boldsymbol{c}_i\cdot\boldsymbol{u}\boldsymbol{\xi}_i\right) \\
&= \boldsymbol{q} + \frac{3}{2}p\boldsymbol{u} + \frac{1}{2}\rho\boldsymbol{u}^2\boldsymbol{u} + m\sum_{i=1}^{b} N_i(\boldsymbol{\xi}_i - \boldsymbol{u})\cdot\boldsymbol{u}\boldsymbol{\xi}_i.
\end{aligned}$$

The last term is equal to $\boldsymbol{u}\cdot\mathsf{P}$ because

$$\boldsymbol{u}\cdot[(\boldsymbol{\xi}_i - \boldsymbol{u}) \otimes (\boldsymbol{\xi}_i - \boldsymbol{u})] = (\boldsymbol{\xi}_i - \boldsymbol{u})[\boldsymbol{u}\cdot(\boldsymbol{\xi}_i - \boldsymbol{u})]$$

and

$$\sum_{i=1}^{b} N_i\boldsymbol{u}\boldsymbol{u}\cdot(\boldsymbol{\xi}_i - \boldsymbol{u}) = 0.$$

We therefore obtain the conservation law:

$$\partial_t\left(\frac{3}{2}p + \frac{1}{2}\rho\boldsymbol{u}^2\right) + \boldsymbol{\nabla}\cdot\left(\frac{3}{2}p\boldsymbol{u} + \frac{1}{2}\rho\boldsymbol{u}^2\boldsymbol{u} + \boldsymbol{u}\cdot\mathsf{P} + \boldsymbol{q}\right) = 0. \qquad (1.3.2\text{-}6)$$

Chapter 2

THE MAXWELLIAN STATE

As in classical kinetic theory, by introducing a properly defined H-Boltzmann function, we can prove that for a gas in a uniform state, the distribution of velocities tends to a distribution, called Maxwellian, in which each collision brings no contribution to the evolution of densities. Among all distributions of velocities which correspond to given state variables, one and only one is the Maxwellian, and the corresponding H-function is minimal. When the velocity distribution is Maxwellian, the evolution of the gas is governed by the Euler equations (which form a hyperbolic system) and by the associated shock equations.

2.1 The *H*-Boltzmann Theorem

2.1.1 *H*-Boltzmann Theorem

For a spatially homogeneous gas with $N_i = N_i(t)$, the H-Boltzmann function

$$H = \sum_{i=1}^{b} N_i \ln(N_i) \tag{2.1.1-1}$$

is a function of time, the derivative of which is

$$\frac{dH}{dt} = \sum_{i=1}^{b} \{1 + \ln(N_i)\} \frac{dN_i}{dt}.$$

We will denote by $\ln(\mathbf{N})$ the vector in $\mathbb{R}^b$ with components $\{\ln(N_i)\}$. By using the kinetic equations (1.1.2-1) and the formula (1.1.3-2) we can write:

$$\begin{aligned}\frac{dH}{dt} &= \frac{1}{8}\sum_{i,j,k,l}\{1+\ln(N_i)+1+\ln(N_j)-1-\ln(N_k)-1-\ln(N_l)\}\,A_{ij}^{kl}(N_kN_l-N_iN_j)\\ &= \frac{1}{8}\sum_{i,j,k,l}\ln\left(\frac{N_iN_j}{N_kN_l}\right)\left(1-\frac{N_iN_j}{N_kN_l}\right)A_{ij}^{kl}N_kN_l. \qquad (2.1.1\text{-}2)\end{aligned}$$

The transition probabilities A_{ij}^{kl} and the densities N_kN_l are non-negative. Also the function $(1-x)\ln(x)$, defined for $x>0$, is non-positive (zero only if $x=1$). Thus we have:

$$\frac{dH}{dt} \leq 0$$

with

$$\frac{dH}{dt} = 0 \quad \text{if and only if} \quad N_kN_l = N_iN_j.$$

Thus we have the following theorem:

Theorem 2.1 The H-Boltzmann function is a non-increasing function of time. As a consequence the evolution of the gas is an irreversible process. The H-function cannot decrease indefinitely as time increases because:

$$N_i\ln(N_i) \geq -\frac{1}{e}, \quad \text{and therefore} \quad H \geq -\frac{b}{e}.$$

Therefore the function H tends to a limit value H^* which corresponds to an equilibrium state for which $\frac{dH}{dt}=0$ and so:

$$\ln(N_i)+\ln(N_j)-\ln(N_k)-\ln(N_l)=0. \qquad (2.1.1\text{-}3)$$

2.1.2 Maxwellian State

As in classical kinetic theory, the limiting equilibrium state in which the densities $\{N_i\}$ satisfy (2.1.1-3) is called Maxwellian. The following three properties are equivalent:

(a) State is Maxwellian (so that condition (2.1.1-3) holds);

(b) $\mathbf{F}(\mathbf{N},\,\mathbf{N}) = \mathbf{0}$;

(c) $\langle\ln(\mathbf{N}),\,\mathbf{F}(\mathbf{N},\,\mathbf{N})\rangle = 0$.

It is easy to see that (a) $\to$ (b) $\to$ (c). The fact that (c) $\to$ (a) is a consequence of the H-Boltzmann theorem. Indeed we have:

$$\langle \ln(\mathbf{N}),\, \mathbf{F}(\mathbf{N},\, \mathbf{N})\rangle = \frac{1}{8}\sum_{i,\,j,\,k,\,l} \ln\left(\frac{N_i N_j}{N_k N_l}\right)\left(1-\frac{N_i N_j}{N_k N_l}\right) A_{ij}^{kl} N_k N_l,$$

which is zero only when $N_i N_j = N_k N_l$ which is precisely (2.1.1-3).

By (2.1.1-3) a gas in a Maxwellian state is characterized by the fact that $\ln(\mathbf{N})$ belongs to $\mathbb{F}$, so that there are s coefficients c_i such that

$$\ln(\mathbf{N}) = c_1\mathbf{V}^{(1)} + c_2\mathbf{V}^{(2)} + \ldots + c_s\mathbf{V}^{(s)}.$$

The components a_i and b_j of the vector $\mathbf{N}$ can also be expressed in terms of the coefficients c_i. If the components of $\mathbf{V}^{(i)}$ and $\mathbf{W}^{(j)}$ are denoted by $V_l^{(i)}$ and $W_l^{(j)}$, respectively, then we have:

$$\left.\begin{aligned} a_i &= \langle \mathbf{N},\, \mathbf{V}^{(i)}\rangle = \sum_{l=1}^{b} V_l^{(i)} \exp\left(\sum_{k=1}^{s} c_k V_l^{(k)}\right), && i = 1,\, 2,\, \ldots,\, s,\\ b_j &= \langle \mathbf{N},\, \mathbf{W}^{(j)}\rangle = \sum_{l=1}^{b} W_l^{(j)} \exp\left(\sum_{k=1}^{s} c_k V_l^{(k)}\right), && j = s+1,\, s+2,\, \ldots,\, b.\end{aligned}\right\} \quad (2.1.2\text{-}1)$$

A Maxwellian state is completely determined by the knowledge of the a_i because the functional determinant $J = \left|\dfrac{\partial a_i}{\partial c_r}\right|$ is different from zero. Indeed:

$$\frac{\partial a_i}{\partial c_r} = \sum_{l=1}^{b} V_l^{(i)} V_l^{(r)} \exp\left(\sum_{k=1}^{s} c_k V_l^{(k)}\right) = \sum_{l=1}^{b} V_l^{(i)} V_l^{(r)} N_l,$$

and

$$\sum_{i=1}^{s}\sum_{r=1}^{s} x_i x_r \frac{\partial a_i}{\partial c_r} = \sum_{l=1}^{b} N_l \left(\sum_{i=1}^{s} x_i V_l^{(i)}\right)\left(\sum_{r=1}^{s} x_r V_l^{(r)}\right) = \sum_{l=1}^{b} N_l \left(\sum_{i=1}^{s} x_i V_l^{(i)}\right)^2 \geq 0.$$

If the densities $\{N_l\}$ are all assumed to be positive then we can have equality only if $\sum_{i=0}^{s} x_i V_l^{(i)} = 0$. That means $x_i = 0$ for all i because the vectors $\mathbf{V}^{(i)}$ are independent. Thus the quadratic form

$$\sum_{i=1}^{s}\sum_{r=1}^{s} x_i x_r \frac{\partial a_i}{\partial c_r}$$

is positive definite which in turn implies that the functional determinant J is not zero.

Theorem 2.2 If the densities $\mathbf{N}^{(1)}$ and $\mathbf{N}^{(2)}$ corresponding to two Maxwellian states possess the same components a_i on the space $\mathbb{F}$ of summational invariants (this means $\langle \mathbf{N}^{(1)} - \mathbf{N}^{(2)}, \mathbf{V}^{(i)} \rangle = 0$ $\forall\ i$), then $\mathbf{N}^{(1)} = \mathbf{N}^{(2)}$.

Theorem 2.2 is global and is proved in [35]. By the way, given the $\{a_i\}$, the $\{c_i\}$ are uniquely determined locally, because $J \neq 0$, and hence so are the $\{b_j\}$ by Eqs. (2.1.2-1).

It follows from this theorem that to each microscopic state $\mathbf{N}$, there corresponds one and only one Maxwellian state $\mathbf{N}^{(0)}$ such that:

$$\langle \mathbf{N},\, \mathbf{V}^{(i)} \rangle = \langle \mathbf{N}^{(0)},\, \mathbf{V}^{(i)} \rangle, \qquad \forall\ i = 1,\, 2,\, \ldots,\, s.$$

For a gas in a Maxwellian state, the $\{N_i\}$ and $\{b_j\}$ are determined by the $\{a_i\}$. This is analogous to the classical kinetic theory in which the Maxwell distribution function depends only on the state variables p, T and $\boldsymbol{u}$. If the $\{a_i\}$ are functions of both time and position, we say that the Maxwellian state is locally Maxwellian.

2.1.3 Maxwellian State Associated with State Variables

Theorem 2.3 When the macroscopic variables are given, the densities $\{N_i\}$ of the associated Maxwellian state are those for which the H-Boltzmann function is minimum.

Indeed we have:

$$H = \sum_{i=1}^{s} N_i \ln(N_i), \qquad \sum_{i=1}^{s} N_i V_i^{(k)} = a_k, \quad k = 1,\, 2,\, \ldots,\, s.$$

It follows that the N_i which minimize the H function satisfy the following equations:

(1) $dH = \sum_{i=1}^{s} [1 + \ln(N_i)] dN_i = 0$;

(2) with $\sum_{i=1}^{s} V_i^{(k)} dN_i = 0$, $k = 1,\, 2,\, \ldots,\, s$.

From (2) we see that $d\mathbf{N}$ is an arbitrary vector of the space $\mathbb{F}^{\perp}$ orthogonal to $\mathbb{F}$ ($\mathbb{F} \otimes \mathbb{F}^{\perp} = \mathbb{R}^b$). It now follows from (1) that the vector with components $1 + \ln(N_i)$ belongs to $\mathbb{F}$. Because the vector $(1,\, 1,\, \ldots,\, 1)$ belongs to $\mathbb{F}$, we have

$$\ln(\mathbf{N}) \in \mathbb{F},$$

and thus

$$\ln(N_i) = \sum_{k=1}^{s} c_k V_i^{(k)},$$

so the densities $\{N_i\}$ for which $dH = 0$ are those of the Maxwellian state associated with the macroscopic state variables.

It remains only to show that the extremum of the H function with conditions $\sum_{i=1}^{s} N_i V_i^{(k)} = a_k$ really is a minimum. To this end we first note that H is a convex function of the variables $\{N_i\}$ because $x\ln(x)$ is a convex function. Since the N_i are connected by s relations, we can take $(b-s)$ of them as being the independent variables say, N_{s+1}, N_{s+2}, ..., N_b. The remaining $\{N_i\}$, $(i \leq s)$, can be expressed in the form:

$$N_i = \sum_{j=s+1}^{b} b_{ij} N_j, \qquad i \leq s,$$

Let us define:

$$\mathcal{H}(N_{s+1},\, N_{s+2},\, \ldots,\, N_b) = H(\textstyle\sum_{j=s+1}^{b} b_{1j} N_j,\, \sum_{j=s+1}^{b} b_{2j} N_j,\, \ldots,\, N_{s+1},\, \ldots,\, N_b)$$

The function $\mathcal{H}$ so defined is a convex function of the variables N_{s+1}, N_{s+2}, ..., N_b, because:

$$\begin{aligned}
& \mathcal{H}(N_{s+1},\, N_{s+2},\, \ldots,\, N_b) + \mathcal{H}(M_{s+1},\, M_{s+2},\, \ldots,\, M_b) \\
= \;& H(\textstyle\sum_{j=s+1}^{b} b_{1j} N_j,\, \sum_{j=s+1}^{b} b_{2j} N_j,\, \ldots,\, N_{s+1},\, \ldots,\, N_b) \\
& + H(\textstyle\sum_{j=s+1}^{b} b_{1j} M_j,\, \sum_{j=s+1}^{b} b_{2j} M_j,\, \ldots,\, M_{s+1},\, \ldots,\, M_b) \\
\leq \;& 2\mathcal{H}((N_{s+1} + M_{s+1})/2,\, (N_{s+2} + M_{s+2})/2,\, \ldots,\, (N_b + M_b)/2).
\end{aligned}$$

Thus the extremum of the $\mathcal{H}$ function which is the same as the extremum of the H function subject to the conditions $\sum_{i=1}^{s} N_i V_i^{(k)} = a_k$, $(k = 1,\, 2,\, \ldots,\, s)$ must indeed be a minimum.

2.2 Euler Equations

2.2.1 System of Euler Equations

From the kinetic equations:

$$\partial_t \mathbf{N} + \mathsf{A} \cdot \mathbf{N} = \mathbf{F}(\mathbf{N},\, \mathbf{N}),$$

we obtained the transport equations, and then as a particular case, the conservation equations (1.3.2-2):

$$\partial_t a_k + \sum_{i=1}^{s} \boldsymbol{L}_{ki}\cdot\boldsymbol{\nabla} a_i + \sum_{j=s+1}^{b} \boldsymbol{M}_{kj}\cdot\boldsymbol{\nabla} b_j = 0.$$

There are s conservation equations (corresponding to $k = 1, 2, \ldots, s$). When the velocity distribution is Maxwellian, the densities can be expressed in terms of the $\{a_i\}$, and hence the $b_j = \langle \mathbf{W}^{(j)}, \mathbf{N}\rangle$ are also known in terms of the $\{a_i\}$. In this case the conservation equations become a closed system of s equations relating the s unknown functions a_i ($i = 1, 2, \ldots, s$).

The Euler equations can be written in a simple form when we take as unknown functions not the $\{a_i\}$, but the functions $\{c_l\}$ determined by:

$$\ln(\mathbf{N}) = c_1\mathbf{V}^{(1)} + c_2\mathbf{V}^{(2)} + \ldots + c_s\mathbf{V}^{(s)},$$

or

$$N_i = \exp\left(\sum_{l=1}^{s} c_l V_i^{(l)}\right),$$

we then obtain:

$$\begin{aligned}
&\frac{\partial N_i}{\partial c_k} = V_i^{(k)} N_i,\\
&\langle \mathbf{N},\, \mathbf{V}^{(k)}\rangle = \sum_{i=1}^{b} N_i V_i^{(k)} = \sum_{i=1}^{b} \frac{\partial N_i}{\partial c_k},\\
&\langle \mathsf{A}\cdot\mathbf{N},\, \mathbf{V}^{(k)}\rangle = \sum_{i=1}^{b} (\boldsymbol{\xi}_i\cdot\boldsymbol{\nabla} N_i) V_i^{(k)} = \sum_{i=1}^{b} \boldsymbol{\nabla}\cdot\frac{\partial}{\partial c_k}\left(\boldsymbol{\xi}_i N_i\right).
\end{aligned}$$

The conservation equation for the vector $\mathbf{V}^{(k)}$ may therefore be written:

$$\frac{\partial}{\partial t}\frac{\partial L}{\partial c_k} + \boldsymbol{\nabla}\cdot\frac{\partial \boldsymbol{M}}{\partial c_k} = 0, \qquad k = 1, 2, \ldots, s, \tag{2.2.1-1}$$

where

$$L = \sum_{i=1}^{b} N_i = n,$$

$$\boldsymbol{M} = \sum_{i=1}^{b} \boldsymbol{\xi}_i N_i = n\boldsymbol{u}.$$

Equations (2.2.1-1) are the system of Euler equations. We can see that they are symmetric by writing the derivatives explicitly:

$$\sum_{l=1}^{s} \frac{\partial^2 L}{\partial c_k \partial c_l}\frac{\partial c_l}{\partial t} + \sum_{l=1}^{s} \frac{\partial^2 \boldsymbol{M}}{\partial c_k \partial c_l}\cdot\boldsymbol{\nabla} c_l = 0, \qquad k = 1, 2, \ldots, s. \tag{2.2.1-2}$$

2.2.2 Characteristic Velocities

The evolution of a gas in a Maxwellian state is governed by the Euler equations. Small perturbations, acoustic waves for example, propagate with velocities $\boldsymbol{\xi}^*$ equal to the characteristic velocities of these equations. The matrices

$$\mathsf{L} = \left(\frac{\partial^2 L}{\partial c_k \partial c_l}\right) \qquad \text{and} \qquad \mathsf{M} = \left(\frac{\partial^2 \boldsymbol{M}}{\partial c_k \partial c_l}\right)$$

are symmetric and the matrix L is positive definite because as we have shown in Sec. 2.1.2:

$$\sum_{k=1}^{b}\sum_{l=1}^{b} x_k x_l \frac{\partial^2 L}{\partial c_k \partial c_l} = \sum_{k=1}^{b} N_k \left(\sum_{l=1}^{s} x_l V_k^{(l)}\right)^2 \geq 0.$$

It follows that the characteristic speeds in the direction of the unit vector $\widehat{\boldsymbol{n}}$, speeds which are the eigenvalues of the matrix $\mathsf{L}^{-1}\mathsf{M} \cdot \widehat{\boldsymbol{n}}$, or the roots of the equation:

$$\det(\xi^* \mathsf{L} - \mathsf{M} \cdot \widehat{\boldsymbol{n}}) = 0, \qquad (2.2.2\text{-}1)$$

are all real. The elements of the matrix $\mathsf{M} \cdot \widehat{\boldsymbol{n}}$ are $(\partial^2 \boldsymbol{M}/\partial c_k \partial c_l) \cdot \widehat{\boldsymbol{n}}$.

2.2.3 Shock Wave Equations

Consider a physical law written in integral form:

$$\frac{d}{dt}\int_V \phi \, d\boldsymbol{r} = \int_S \boldsymbol{A} \cdot \widehat{\boldsymbol{n}} \, dS, \qquad (2.2.3\text{-}1)$$

where ϕ and $\boldsymbol{A}$ denote characteristic quanlities of the medium, the evolution of which we are studying. The volume V and the surface S are assumed to be material (that is they are always composed of the same molecules which do not mix). The unit vector $\widehat{\boldsymbol{n}}$ is the outward directed normal to S. Since S is a material surface, its local displacement velocity is $\boldsymbol{V} \cdot \widehat{\boldsymbol{n}}$, where $\boldsymbol{V}$ is the gas velocity.

From relation (2.2.3-1) we can deduce the following equation for continuously differentiable functions ϕ and $\boldsymbol{A}$:

$$\partial_t \phi + \boldsymbol{\nabla} \cdot (\boldsymbol{A} + \phi \boldsymbol{V}) = 0. \qquad (2.2.3\text{-}2)$$

When the functions ϕ and $\boldsymbol{A}$ are discontinuous across a surface Σ (called a shock wave) which moves with velocity ξ in the direction of the unit vector $\widehat{\boldsymbol{n}}$, we deduce from relation (2.2.3-1) the shock condition:

$$[\phi\,(\xi - \boldsymbol{V} \cdot \widehat{\boldsymbol{n}}) - \boldsymbol{A} \cdot \widehat{\boldsymbol{n}}] = 0, \qquad (2.2.3\text{-}3)$$

where $[Q]$ is the jump of Q across the shock wave. By setting $\boldsymbol{B} = \boldsymbol{A} + \phi \boldsymbol{V}$, we can see that the shock condition corresponding to the conservation equation:

$$\partial_t \phi + \boldsymbol{\nabla} \cdot \boldsymbol{B} = 0$$

is

$$[\phi \xi - \boldsymbol{B} \cdot \widehat{\boldsymbol{n}}] = 0.$$

Applying this result to the Euler equations, we see that the shock equations for the Euler equations written in the form of Eqs. (2.2.1-1) are

$$\left[\frac{\partial L}{\partial c_k} \xi - \widehat{\boldsymbol{n}} \cdot \frac{\partial \boldsymbol{M}}{\partial c_k} \right] = 0, \qquad k = 1, 2, \ldots, s. \tag{2.2.3-4}$$

A characteristic surface moving with velocity $\widehat{\boldsymbol{n}} \xi$ will move away from the shock wave and will be a diverging wave if the difference $(\xi^* - \xi)$ is negative when we are in front of the shock and positive when we are behind the shock. The Lax criterion [48] states that a shock is stable when the number of diverging waves is equal to $(s - 1)$, and unstable when this number differs from $(s - 1)$.

A second criterion of stability is to consider as stable as any shock which can be obtained as a limit of a continuous flow. To obtain a criterion from this point of view, we multiply both sides of the kinetic equation (1.1.2-1) by $1 + \ln(N_i)$ and sum over the index i:

$$\begin{aligned} &\frac{\partial}{\partial t} \left(\sum_{i=1}^{b} N_i \ln(N_i) \right) + \boldsymbol{\nabla} \cdot \left(\sum_{i=1}^{b} \boldsymbol{\xi}_i N_i \ln(N_i) \right) \\ &\qquad = \frac{1}{2} \sum_{i=1}^{b} (1 + \ln(N_i) \sum_{j,k,l} A_{ij}^{kl} (N_k N_l - N_i N_j). \end{aligned} \tag{2.2.3-5}$$

As we have seen in the proof of the H-theorem, the right hand side of this equation is negative or zero. By invoking the same limiting process used to establish the shock equations, we find:

$$\left[\sum_{i=1}^{b} (\xi - \boldsymbol{\xi}_i \cdot \widehat{\boldsymbol{n}}) N_i \ln(N_i) \right] \leq 0, \tag{2.2.3-6}$$

and a shock will be stable if through the shock the function

$$H_1 = \sum_{i=1}^{b} (\xi - \boldsymbol{\xi}_i \cdot \widehat{\boldsymbol{n}}) N_i \ln N_i \tag{2.2.3-7}$$

is decreasing.

Chapter 3

STUDY OF SOME PARTICULAR MODELS

Some particular models are discussed in the literature. A model with two velocities is presented by Carleman [27], but this model does not require conservation of momentum during collisions. Some regular plane models with velocities of the same modulus are better, and an example with four velocities is studied in Section 3.1. Three-dimensional models have been introduced by Broadwell involving velocities of equal modulus. One model has six velocities [10] and the other eight [11], and these models are discussed in Sec. 3.2 and 3.3, respectively. As a consequence of the fact that the velocities have the same modulus, both of Broadwell's models suffer the inconvenience of having the temperature being a function of the velocity. In order to allow the temperature to be an independent variable, in Section 3.4 we introduce a three-dimensional model with 14 velocities. This model is a combination of models with six and eight velocities. Of course there are many possible generalizations.

3.1 Regular Plane Model with Four Velocities

3.1.1 Kinetic Equations

The model considered in this section is a plane model. In the xy plane, the molecules can only have one of the following four vectors as a velocity:

$$\boldsymbol{\xi}_1 = c\,(1,\,0), \qquad \boldsymbol{\xi}_2 = c\,(0,\,1), \qquad \boldsymbol{\xi}_3 = c\,(-1,\,0), \qquad \boldsymbol{\xi}_4 = c\,(0,\,-1).$$

The densities $N_i(x,\,y,\,t)$ are independent of the third space variable z. The only non-trivial collisions are:

$$(\boldsymbol{\xi}_1,\,\boldsymbol{\xi}_3) \longleftrightarrow (\boldsymbol{\xi}_2,\,\boldsymbol{\xi}_4).$$

Therefore $a_{12}^{34} = \frac{1}{2}$ and $A_{12}^{34} = \frac{1}{2}2cS$. The kinetic equations are:

$$\left.\begin{aligned}
\partial_t N_1 + c\partial_x N_1 &= cS(N_2N_4 - N_1N_3), \\
\partial_t N_2 + c\partial_y N_2 &= cS(N_1N_3 - N_2N_4), \\
\partial_t N_3 - c\partial_x N_3 &= cS(N_2N_4 - N_1N_3), \\
\partial_t N_4 - c\partial_y N_4 &= cS(N_1N_3 - N_2N_4).
\end{aligned}\right\} \tag{3.1.1-1}$$

The summational invariants are the vectors $\mathbf{\Phi} = (\phi_1,\, \phi_2,\, \phi_3,\, \phi_4)$ in $\mathbb{R}^4$ satisfying:

$$\phi_1 + \phi_3 - \phi_2 - \phi_4 = 0. \tag{3.1.1-2}$$

Since only ϕ_1, ϕ_2 and ϕ_3 can be chosen arbitrarily, the dimension of $\mathbb{F}$ is $s = 3$. A basis of $\mathbb{F}$ is:

$$\begin{aligned}
\mathbf{V}^{(1)} &= \frac{1}{2}\,(1,\, 1,\, 1,\, 1)\,, \\
\mathbf{V}^{(2)} &= \frac{1}{\sqrt{2}}\,(1,\, 0,\, -1,\, 0)\,, \\
\mathbf{V}^{(3)} &= \frac{1}{\sqrt{2}}\,(0,\, 1,\, 0,\, -1)\,.
\end{aligned}$$

These vectors correspond to the conservation of mass and the components of momentum.

After the densities have been computed, we calculate the macroscopic variables using the relations:

$$\begin{aligned}
n &= (N_1 + N_2 + N_3 + N_4), \\
nu &= c\,(N_1 - N_3), \\
nv &= c\,(N_2 - N_4),
\end{aligned} \tag{3.1.1-3}$$

and

$$\begin{aligned}
\mathsf{P} &= m\sum_{i=1}^{4} N_i(\boldsymbol{\xi}_i - \boldsymbol{u})(\boldsymbol{\xi}_i - \boldsymbol{u}) \\
&= mc^2 \begin{pmatrix} N_1 + N_3 & 0 \\ 0 & N_2 + N_4 \end{pmatrix} - mn \begin{pmatrix} u^2 & uv \\ uv & v^2 \end{pmatrix} \\
p &= nk_BT = \frac{1}{2}mn\,[c^2 - (u^2 + v^2)].
\end{aligned}$$

3.1.2 The Euler Equations

In the Maxwellian state the densities depend on the three functions c_1, c_2 and c_3:

$$\left.\begin{aligned} N_1 &= \exp\left(\frac{c_1}{2} + \frac{c_2}{\sqrt{2}}\right), \\ N_2 &= \exp\left(\frac{c_1}{2} + \frac{c_3}{\sqrt{2}}\right), \\ N_3 &= \exp\left(\frac{c_1}{2} - \frac{c_2}{\sqrt{2}}\right), \\ N_4 &= \exp\left(\frac{c_1}{2} - \frac{c_3}{\sqrt{2}}\right). \end{aligned}\right\} \tag{3.1.2-1}$$

From the general theory the Euler equations are:

$$\frac{\partial}{\partial t}\left(\frac{\partial n}{\partial c_k}\right) + \boldsymbol{\nabla} \cdot \left(\frac{\partial n\boldsymbol{u}}{\partial c_k}\right) = 0, \tag{3.1.2-2}$$

which in our case gives:

$$\left.\begin{aligned} &\frac{\partial n}{\partial t} + \frac{\partial nu}{\partial x} + \frac{\partial nv}{\partial y} = 0, \\ &\frac{\partial nu}{\partial t} + \frac{\partial}{\partial x}\left\{n\frac{(c^2 + u^2 - v^2)}{2}\right\}, \\ &\frac{\partial nv}{\partial t} + \frac{\partial}{\partial y}\left\{n\frac{(c^2 - u^2 + v^2)}{2}\right\}. \end{aligned}\right\} \tag{3.1.2-3}$$

When the gas is at rest (*i.e.*, $u = v = 0$), the characteristic speeds in all directions have the same values, namely 0, $\pm c/\sqrt{2}$.

3.1.3 The $\boldsymbol{H}$-Boltzmann Function

The first derivative of the H-Boltzmann function is negative. It is interesting to note that for the regular plane four velocities model, it is true that the successive derivatives of the H-Boltzmann function alternate in sign [45]:

$$(-1)^k \frac{d^k H}{dt^k} \geq 0, \qquad k = 1, 2, \ldots. \tag{3.1.3-1}$$

As a consequence of the first Euler equation, when the densities are independent of the space variables, the total density n is a constant. Letting $n_i = N_i/n$ and $\tau = cSnt$, we can write the kinetic equations (3.1.1-1) as:

$$\frac{dn_i}{d\tau} = n_{i+1}n_{i+3} - n_i n_{i+2}, \qquad i = 1, 2, 3, 4, \quad \text{with } (n_1 + n_2 + n_3 + n_4) = 1. \tag{3.1.3-2}$$

In the above equation we are considering $n_k = n_l$ when $k \equiv l \pmod 4$. From equations (3.1.3-2) we deduce:

$$\frac{d^k n_i}{d\tau^k} = (-1)^{k+1}\frac{dn_i}{d\tau}, \qquad i = 1,\, 2,\, 3,\, 4. \tag{3.1.3-3}$$

The H-Boltzmann function is:

$$H = \sum_{i=1}^{4} N_i \ln(N_i) = n\ln(n) + n\sum_{i=1}^{4} n_i \ln(n_i),$$

and because n is a positive constant, the derivatives with respect to t of H have the same sign as the derivatives with respect to τ of:

$$h(\tau) = \sum_{i=1}^{4} n_i(\tau)\ln(n_i(\tau)). \tag{3.1.3-4}$$

By taking successive derivatives we obtain:

$$\begin{aligned}
\frac{dh}{d\tau} &= \sum_{i=1}^{4} \ln(n_i)\frac{dn_i}{d\tau} = (n_1 n_3 - n_2 n_4)\ln\left(\frac{n_2 n_4}{n_1 n_3}\right) \le 0,\\
\frac{d^2 h}{d\tau^2} &= \sum_{i=1}^{4}\left\{\ln(n_i)\frac{d^2 n_i}{d\tau^2} + \frac{1}{n_i}\left(\frac{dn_i}{d\tau}\right)^2\right\}\\
&= -\frac{dh}{d\tau} + \sum_{i=1}^{4} A_i, \qquad A_i := \frac{1}{n_i}\left(\frac{dn_i}{d\tau}\right)^2\\
\frac{d^{k+2}h}{d\tau^{k+2}} &= -\frac{d^{k+1}h}{d\tau^{k+1}} + \frac{d^k A}{d\tau^k}, \qquad \text{with } A := \sum_{i=1}^{4} A_i.
\end{aligned}$$

The initial values of the densities $\{N_i\}$ are positive, and so is the initial value of A and the derivative $\dfrac{d^2 h}{d\tau^2}$.

To complete the proof of inequalities (3.1.3-1) it suffices to show that:

$$(-1)^k \frac{d^k A_i}{d\tau^k} \ge 0. \tag{3.1.3-5}$$

This will certainly be true if we can show:

$$(-1)^k \frac{d^k A_i}{d\tau^k} \ge A_i \quad \forall\, k, \tag{3.1.3-6}$$

because $A_i \ge 0$. The above inequality can be proved by induction.

For $k = 1$ we have:

$$-\frac{dA_i}{d\tau} = \frac{1}{n_i}\left\{2\left(\frac{dn_i}{d\tau}\right)^2 + A_i\frac{dn_i}{d\tau}\right\} = A_i\left\{2 + \frac{1}{n_i}\frac{dn_i}{d\tau}\right\}.$$

Equation (3.1.3-2) can be written as:

$$n_i + \frac{dn_i}{d\tau} = n_{i+1}n_{i+3} + n_i(n_{i-1} + n_i + n_{i+1}) \geq 0, \tag{3.1.3-7}$$

which proves inequality (3.1.3-6) for $k = 1$. To compute $\frac{d^k A_i}{d\tau^k}$, we differentiate the product $n_i A_i$ in two different ways. First we use formula (3.1.3-3) and then we use Leibniz rule:

$$\begin{aligned}
\frac{d^k(n_i A_i)}{d\tau^k} &= \frac{d^k}{d\tau^k}\left(\frac{dn_i}{d\tau}\right)^2 = (-2)^k \left(\frac{dn_i}{d\tau}\right)^2 \\
\frac{d^k(n_i A_i)}{d\tau^k} &= \sum_{j=0}^{k-1} C_k^j \frac{d^{k-j}n_i}{d\tau^{k-j}} \frac{d^j A_i}{d\tau^j} + n_i \frac{d^k A_i}{d\tau^k},
\end{aligned}$$

where $C_k^j := k!/j!(k-j)!$ is the binomial coefficient. Comparing the last two equations yields:

$$(-1)^k \frac{d^k A_i}{d\tau^k} = \frac{1}{n_i}\left\{2^k \left(\frac{dn_i}{d\tau}\right)^2 + \sum_{j=0}^{k-1}(-1)^j C_k^j \frac{d^j A_i}{d\tau^j}\frac{dn_i}{d\tau}\right\}. \tag{3.1.3-8}$$

We have shown inequality (3.1.3-6) holds for $k = 1$, assume that it holds for $(k-1)$, then the above equality leads to:

$$\begin{aligned}
(-1)^k \frac{d^k A_i}{d\tau^k} &\geq A_i \left\{2^k + \frac{1}{n_i}\frac{dn_i}{d\tau}\sum_{j=0}^{k-1} C_k^j\right\} \\
&= A_i \left\{2^k + (2^k - 1)\frac{1}{n_i}\frac{dn_i}{d\tau}\right\} \\
&= A_i \left\{1 + (2^k - 1)\frac{1}{n_i}\left(n_i + \frac{dn_i}{d\tau}\right)\right\} \\
&\geq A_i.
\end{aligned} \tag{3.1.3-9}$$

This completes the proof of inequality (3.1.3-6), and hence forth inequality (3.1.3-1).

The densities $\{N_i(t)\}$ are monotonic functions of time, and if the initial state is Maxwellian so that $(\bar{n}_1\bar{n}_3 - \bar{n}_2\bar{n}_4) = 0$, then the $\{N_i(t)\}$ are constants.

3.2 Regular Space Model with Six Velocities

3.2.1 Kinetic Equations

The regular space models are related to the regular polyhedrons. There are only five regular convex polyhedrons, the simplest of which is the cube. The simplest models for discrete velocity distributions are related to the symmetries of the cube. In this section we assume that the velocities $\boldsymbol{\xi}_i$ are the six vectors joining the center of the cube to the centers of the faces. These velocities are:

$$\boldsymbol{\xi}_1 = c\,(1,\,0,\,0), \qquad \boldsymbol{\xi}_2 = c\,(0,\,1,\,0), \qquad \boldsymbol{\xi}_3 = c\,(0,\,0,\,1),$$

and $\boldsymbol{\xi}_{i+3} = -\boldsymbol{\xi}_i$, $i = 1,\,2,\,3$.

The densities $\{N_i(x,y,z,t)\}$ now depend on all the space variables. The only nontrivial collisions are:

$$(\boldsymbol{\xi}_1,\,\boldsymbol{\xi}_4) \longleftrightarrow (\boldsymbol{\xi}_2,\,\boldsymbol{\xi}_5) \longleftrightarrow (\boldsymbol{\xi}_3,\,\boldsymbol{\xi}_6).$$

Therefore:

$$\begin{aligned} a_{14}^{25} = a_{14}^{36} = a_{25}^{36} &= \frac{1}{3}, \\ A_{14}^{25} = A_{14}^{36} = A_{25}^{36} &= \frac{2}{3}cS. \end{aligned}$$

Putting $N_k = N_l$ if $k = l \pmod 6$, the kinetic equations are:

$$\partial_t N_i + \boldsymbol{\xi}_i \cdot \boldsymbol{\nabla} N_i = \frac{2cS}{3}\left(N_{i+1}N_{i+4} + N_{i+2}N_{i+5} - 2N_iN_{i+3}\right), \qquad i = 1,\,2,\,\ldots,\,6. \quad (3.2.1\text{-}1)$$

The summational invariants are the vectors $\boldsymbol{\Phi}$ in $\mathbb{R}^6$ satisfying the equations:

$$\left.\begin{aligned} \phi_1 + \phi_4 - \phi_2 - \phi_5 &= 0, \\ \phi_1 + \phi_4 - \phi_3 - \phi_6 &= 0. \end{aligned}\right\} \quad (3.2.1\text{-}2)$$

Thus, four of the components ϕ_i can be chosen arbitrarily, which means that the dimension of the space $\mathbb{F}$ is $s = 4$.

3.2.2 The Euler Equations

We will limit ourselves to the case where the densities $\{N_i\}$ are independent of y and z, and where the densities N_2, N_5, N_3 and N_6 are equal. In this case, we have $s = 2$, and from the kinetic equations we can obtain for example the following two conservation equations:

$$\left.\begin{aligned} \partial_t(N_1 + N_4 - 2N_2) + c\partial_x(N_1 - N_4) &= 0, \\ \partial_t(N_1 - N_4) + c\partial_x(N_1 + N_4) &= 0. \end{aligned}\right\} \quad (3.2.2\text{-}1)$$

The macroscopic variables are:

$$\left.\begin{aligned} n &= N_1 + N_4 + 4N_2, \\ nu &= c(N_1 - N_4). \end{aligned}\right\} \tag{3.2.2-2}$$

When the gas is in a Maxwellian state:

$$N_1 N_4 - N_2^2 = 0. \tag{3.2.2-3}$$

We can express N_1, N_2 and N_4 in terms of n and u. In particular:

$$N_1 + N_4 = \frac{n}{3}\left(-1 + 2\sqrt{1 + 3\frac{u^2}{c^2}}\right), \tag{3.2.2-4}$$

which enables us to write the Euler equations in the form:

$$\left.\begin{aligned} &\frac{\partial n}{\partial t} + \frac{\partial nu}{\partial x} = 0, \\ &\frac{\partial nu}{\partial t} + \frac{\partial}{\partial x}\left\{\frac{nc}{3}\left(-c + 2\sqrt{c^2 + 3u^2}\right)\right\}. \end{aligned}\right\} \tag{3.2.2-5}$$

If we return to the general case considered in Section 3.2.1, it can be seen that the characteristic speeds have the same values in all directions, namely 0, $\pm c/\sqrt{3}$.

3.3 Regular Space Model with Eight Velocities

In this Section we assume that the velocities $\{\boldsymbol{\xi}_i\}$ are the eight vectors joining the center of a cube with its vertices [11]. The components of the velocities are:

$$\boldsymbol{\xi}_1 = c\,(-1,\,1,\,1), \qquad \boldsymbol{\xi}_2 = c\,(1,\,1,\,1), \qquad \boldsymbol{\xi}_3 = c\,(-1,\,-1,\,1), \qquad \boldsymbol{\xi}_4 = c\,(1,\,-1,\,1),$$

and $\boldsymbol{\xi}_i = \boldsymbol{\xi}_{9-i}$, $i = 5, 6, 7, 8$. The nontrivial collisions are of two types:

$$(\boldsymbol{\xi}_1,\,\boldsymbol{\xi}_8) \longleftrightarrow (\boldsymbol{\xi}_2,\,\boldsymbol{\xi}_7) \longleftrightarrow (\boldsymbol{\xi}_3,\,\boldsymbol{\xi}_6) \longleftrightarrow (\boldsymbol{\xi}_4,\,\boldsymbol{\xi}_5),$$

and

$$(\boldsymbol{\xi}_1,\,\boldsymbol{\xi}_4) \longleftrightarrow (\boldsymbol{\xi}_2,\,\boldsymbol{\xi}_3), \qquad (\boldsymbol{\xi}_1,\,\boldsymbol{\xi}_6) \longleftrightarrow (\boldsymbol{\xi}_2,\,\boldsymbol{\xi}_5), \qquad \ldots\,. \tag{3.3-1}$$

There are six collisions of the first type and six of the second type. We also have:

$$a_{18}^{27} = a_{18}^{36} = \ldots = \frac{1}{4},$$
$$a_{14}^{23} = a_{16}^{25} = \ldots = \frac{1}{2},$$
$$A_{18}^{27} = A_{18}^{36} = \ldots = \frac{\sqrt{3}}{2}\, cS,$$
$$A_{14}^{23} = A_{16}^{25} = \ldots = \sqrt{2}\, cS.$$

The first of the eight kinetic equations is:

$$\begin{aligned} &\partial_t N_1 - c\partial_x N_1 + c\partial_y N_1 + c\partial_z N_1 \\ &= \frac{\sqrt{3}}{2} cS \left(N_2N_7 + N_3N_6 + N_4N_5 - 2N_1N_8\right) \\ &\quad + \sqrt{2} cS \left(N_2N_3 + N_3N_5 + N_5N_2 - N_1(N_4 + N_6 + N_7)\right). \end{aligned} \tag{3.3-2}$$

The dimension of the space $\mathbb{F}$ is $s = 4$.

When the densities are independent of z and satisfy the relation $N_{i+4} = N_i$, then this case is similar to the regular plane model except that the velocities are now parallel to the angle bisectors of the coordinate axes instead of being parallel to the axes themselves as in Section 3.1.

The Euler equations for this simple case are:

$$\left.\begin{aligned} &\frac{\partial n}{\partial t} + \frac{\partial nu}{\partial x} + \frac{\partial nv}{\partial y} = 0, \\ &\frac{\partial nu}{\partial t} + \frac{\partial nc^2}{\partial x} + \frac{\partial nuv}{\partial y} = 0, \\ &\frac{\partial nv}{\partial t} + \frac{\partial nuv}{\partial x} + \frac{\partial nc^2}{\partial y} = 0. \end{aligned}\right\} \tag{3.3-3}$$

3.4 A Space Model with 14 Velocities

In the models studied in the three previous sections, all the velocities $\{\boldsymbol{\xi}_i\}$ have the same modulus. As a consequence, the temperature is not an independent macroscopic variable, but is a function of the mean velocity. To obtain a model in which the temperature, the mean velocity and the density are independent macroscopic variables, it is necessary to assume that the molecules can have velocities which are not all of the same modulus. One possible way to satisfy this condition is to superimpose the last two models, and to consider a model with 14 velocities: six with modulus c, and eight with modulus $\sqrt{3}c$. We will denote the six velocities with the modulus c by $\{\boldsymbol{\zeta}_i\}$ and their corresponding densities by $\{M_i\}$. The eight velocities with modulus $\sqrt{3}c$ and their densities will be denoted by $\{\boldsymbol{\xi}_i\}$ and $\{N_i\}$, respectively. In this model there are 27 possible collisions:

$$\begin{array}{rl}
\text{6 collisions like} & (\boldsymbol{\xi}_1,\, \boldsymbol{\xi}_8) \longleftrightarrow (\boldsymbol{\xi}_2,\, \boldsymbol{\xi}_7), \\
\text{6 collisions like} & (\boldsymbol{\xi}_1,\, \boldsymbol{\xi}_4) \longleftrightarrow (\boldsymbol{\xi}_2,\, \boldsymbol{\xi}_5), \\
\text{3 collisions like} & (\boldsymbol{\zeta}_1,\, \boldsymbol{\zeta}_4) \longleftrightarrow (\boldsymbol{\zeta}_2,\, \boldsymbol{\zeta}_5), \\
\text{12 collisions like} & (\boldsymbol{\xi}_1,\, \boldsymbol{\zeta}_1) \longleftrightarrow (\boldsymbol{\xi}_2,\, \boldsymbol{\zeta}_4).
\end{array}$$

The last collisions are between molecules having velocities of different moduli. In order to obtain a gas with temperature as an independent variable it is necessary to have collisions of this type, for otherwise the model would represent two gases moving independently. For the four types of collisions listed above, the probabilities and transition probabilities are respectively:

$$\begin{array}{lcccc}
\text{probabilities } a_{ij}^{kl} & \frac{1}{4} & \frac{1}{2} & \frac{1}{3} & \frac{1}{2} \\
\text{transition probabilities } A_{ij}^{kl} & \frac{\sqrt{3}}{2}cS & \sqrt{2}cS & \frac{2}{3}cS & \frac{\sqrt{6}}{2}cS.
\end{array}$$

We can see that the dimension of the space $\mathbb{F}$ is $s = 5$, and that the summational invariants are m, $m\boldsymbol{\xi}_i$ and $\frac{1}{2}m\boldsymbol{\xi}_i^2$, as in classical kinetic theory. Therefore we can hope to have a good model.

To obtain the kinetic equations we have to add terms to the equations already written which represent the collisions of mixed type such as $(\boldsymbol{\xi}_1,\, \boldsymbol{\zeta}_1) \longleftrightarrow (\boldsymbol{\xi}_2,\, \boldsymbol{\zeta}_4)$. To the right hand side of Eq. (3.3-2) we must add:

$$2cS(N_2M_4 + N_3M_2 + N_5M_3 - N_1(M_1 + M_5 + M_6)), \tag{3.4-1}$$

and the first of equations (3.2.1-1) ($i = 1$) has to be replaced by

$$\begin{aligned}
\partial_t M_1 + c\partial_x M_1 = \frac{2}{3}cS(M_2M_5 + M_3M_6 - 2M_1M_4) \\
+cS((N_2 + N_4 + N_6 + N_8)M_4 - (N_1 + N_3 + N_5 + N_7)M_1).
\end{aligned}$$

We can see that if the gas is at rest in a Maxwellian state:

$$\begin{aligned}
N_i &= N_0, & i &= 1,\, 2,\, \ldots,\, 8, \\
M_j &= M_0, & j &= 1,\, 2,\, \ldots,\, 6,
\end{aligned}$$

then the characteristic speeds are independent of the direction of propagation and have values:

$$\xi^* = \pm c\sqrt{\frac{(12N_0 + M_0)}{(12N_0 + 3M_0)}}. \tag{3.4-2}$$

This is a function of the temperature because we can show that:

$$k_B T = \frac{1}{3}mc^2 \left(1 + \frac{8N_0}{4N_0 + 3M_0}\right). \tag{3.4-3}$$

Chapter 4

GLOBAL SOLUTION OF THE DISCRETE BOLTZMANN EQUATION

In the discrete kinetic theory the initial value problem has a local solution. When the local solution is bounded by a number which depends only on the initial values, the solution exists globally. The first global existence theorem of this type was obtained by Nishida and Mimura [55] for a Broadwell gas (three dimensional model with six velocities) [10] with four of the six densities equal. In the following work a similar theorem is proved for a more complex model: a three dimensional model with fourteen velocities obtained by joining the center of a cube first to the center of each face, then to each vertex. The theorem is proved first when the initial densities are small, then, following a method proposed by Tartar and Crandall [62, 63], when the densities are at first periodic and finally when they are bounded. As a starting point certain properties of the local solution are shown to be satisfied.

4.1 Introduction

The discretization of the velocity space in the kinetic theory of gases allows the replacement of the Boltzmann equation, an integro-differential equation, by a system of semi-linear partial differential equations [35]. For this system, called the discrete Boltzmann equation, the initial value problem has a local solution when the initial values are bounded and differentiable. Among the models with discrete repartition of velocities, one of the simplest is the Broadwell model [10] for which the velocities are obtained by joining the center of a cube at the origin of the velocity space to the center of the faces. Using this model and assuming a one-dimensional motion parallel to one of the velocities and equality of the densities of the four velocities orthogonal to that direction, Nishida and Mimura [55] have proved the global existence of the solution of the initial value problem, if the initial values are small in a certain sense. For a similar model, Tartar and Crandall [62, 63] have proved the global existence of the solution which the initial values are no longer small, but first periodic, and then only bounded. The method of Nishida and Mimura [55] to prove the global existence consists of proving that the local solution is bounded by a constant which depends only on the initial values. The bound is obtained by integration of conservation equations over triangles, an edge of each corresponding to the axis of abscissae (initial time), the other edges being characteristics of the discrete Boltzmann equation. The purpose of the present work is to extend the proof and conclusions first of Nishida and Mimura [55], then of Tartar and Crandall [62, 63], to more complex models. The model considered in this Chapter is a three-dimensional model with fourteen velocities obtained by joining the center of a cube at the origin of the velocity space to the centers of the faces and to the vertices [12]. Section 4.2 is devoted to a summary of the properties of the local solution. The subsequent sections are concerned with the global existence theorem when the initial densities, given and depending only on one of the space variables, are small (Sec. 4.3), periodic (Sec. 4.4) and bounded (Sec. 4.5).

4.2 Properties of the Local Solution

The discrete Boltzmann equation is written, in the general case, in the form [35]:

$$\partial_t N_i + \boldsymbol{\xi}_i \cdot \boldsymbol{\nabla} N_i = \frac{1}{2} \sum_{j,\,k,\,l} A_{ij}^{kl} (N_k N_l - N_i N_j), \qquad \text{for } i = 1,\, 2,\, \ldots,\, b. \tag{4.2-1}$$

The unknown functions $\{N_i(\boldsymbol{x},\, t)\}$ denote the densities of different velocities $\boldsymbol{\xi}_i$, represented by b constant vectors $\boldsymbol{\xi}_1,\, \boldsymbol{\xi}_2,\, \ldots,\, \boldsymbol{\xi}_b$. The coefficients A_{ij}^{kl}, the transition probabilities, are non-negative constants; $\boldsymbol{x}$ is the position vector, with components $x,\, y,\, z$, in a Cartesian rectangular system $Oxyz$; t is the time. The Cauchy problem consists of finding a solution of system (4.2-1) which, at the initial time, takes given values

$$N_i(\boldsymbol{x},\, 0) = N_{i0}(\boldsymbol{x}), \qquad i = 1,\, 2,\, \ldots,\, b. \tag{4.2-2}$$

Theorem 4.1 If the functions $\{N_{i0}(\boldsymbol{x})\}$ are continuous and differentiable, there exists a positive constant δ_0 such that, in the interval $0 < t \leq \delta_0$, the problem consisting of Eqs. (4.2-1) and

(4.2-2) has one unique solution.

This theorem assures the existence and uniqueness of a local solution, some properties of which can be studied by a method of successive approximations. We can put:

$$\left.\begin{aligned} &\partial_t N_i^{(n+1)} + \boldsymbol{\xi}_i \cdot \boldsymbol{\nabla} N_i^{(n+1)} = \frac{1}{2} \sum_{j,k,l} A_{ij}^{kl} (N_k^{(n)} N_l^{(n)} - N_i^{(n)} N_j^{(n)}), \\ &N_i^{(n+1)}(\boldsymbol{x}, 0) = N_{i0}(\boldsymbol{x}), \end{aligned}\right\} \tag{4.2-3}$$

$$N_i^{(1)}(\boldsymbol{x}, 0) = N_{i0}(\boldsymbol{x}), \tag{4.2-4}$$

where $N_i^{(n)}$ denotes nth iterative solution of N_i. We deduce from (4.2-3)

$$N_i^{(n+1)}(\boldsymbol{x}, t) = N_{i0}(\boldsymbol{x} - \boldsymbol{\xi}_i t) + \int_0^t h_i^{(n)}(\boldsymbol{x} - \boldsymbol{\xi}_i s)\, ds, \tag{4.2-5}$$

where $h_i^{(n)}$ is the right hand side of equation (4.2-3). Considering in the four dimensional space the point $A = (\boldsymbol{x}_A, t_A)$ and the points $B_i = (\boldsymbol{x}_A - \boldsymbol{\xi}_i t_A, 0)$, we denote by (D_A) the smallest convex domain of the hyper-plane $t = 0$ containing all the points B_i. From the formula (4.2-5) we deduce

Theorem 4.2 The values of the functions $\{N_i(\boldsymbol{x}, t)\}$ at the point A depend only on the initial values $\{N_{i0}(\boldsymbol{x})\}$ in the domain (D_A).

Theorem 4.3 If the initial densities are independent of one of the space variables, the solution of the problem (4.2-1) and (4.2-2) is also independent of this space variable.

Theorem 4.4 If the initial densities are periodic functions, with a period $\boldsymbol{\varpi}$, the solution of the problem (4.2-1) and (4.2-2) is also periodic in $\boldsymbol{x}$ with the period $\boldsymbol{\varpi}$.

The proofs of these theorems are trivial. A more important result is the following theorem.

Theorem 4.5 If the initial densities satisfy the inequalities $0 \leq N_{i0}(\boldsymbol{x}) \leq K_0$, the solution of the problem (4.2-1) and (4.2-2) satisfies the inequalities $N_i(\boldsymbol{x}, t) \geq 0$ for all $\boldsymbol{x} \in \mathbb{R}^3$ and $0 < t \leq \delta_0$.

To prove this problem we introduce the functions

$$\widetilde{N}_i(\boldsymbol{x},\,t) = e^{\lambda t} N_i(\boldsymbol{x},\,t)$$

for all $\boldsymbol{x} \in \mathbb{R}^3$ and $0 < t \leq \delta_0$, in which λ is a positive constant. From equation (4.2-1) we deduce

$$\partial_t \widetilde{N}_i + \boldsymbol{\xi}_i \cdot \boldsymbol{\nabla} \widetilde{N}_i = \lambda \widetilde{N} + \frac{1}{2} e^{-\lambda t} \sum_{j,\,k,\,l} A_{ij}^{kl} (\widetilde{N}_k \widetilde{N}_l - \widetilde{N}_i \widetilde{N}_j), \tag{4.2-6}$$

$$\left.\begin{aligned} \widetilde{N}_i(\boldsymbol{x},\,t) &= N_{i0}(\boldsymbol{x} - \boldsymbol{\xi}_i t) + \int_0^t h_i(\boldsymbol{x} - \boldsymbol{\xi}_i s, t - s)\, ds, \\ h_i &= \frac{1}{2} e^{-\lambda t} \sum_{j,\,k,\,l} A_{ij}^{kl} \widetilde{N}_k \widetilde{N}_l + \widetilde{N}_i \left(\lambda - \frac{1}{2} \sum_{j,\,k,\,l} A_{ij}^{kl} \widetilde{N}_j \right). \end{aligned}\right\} \tag{4.2-7}$$

We denote $A = \frac{1}{2} b^2 \sup\limits_{i,j,k,l} A_{ij}^{kl}$. In the local solution the densities are bounded by a bound B. If we choose $\lambda > AB$, h_i is positive at the initial time, and, by formula (4.2-7), the densities and the h_i are always positive.

Theorem 4.6 If the initial values are continuous and differentiable functions satisfying the inequalities $0 \leq N_{i0}(\boldsymbol{x}) \leq K_0$, then the unique solution of the problem (4.2-1) and (4.2-2) exists for $\boldsymbol{x} \in \mathbb{R}^3$ and $0 < t \leq \delta_0 = (AK_0)^{-1}$.

To prove this last theorem we remark, as a consequence of the positivity of the densities, that the solution of the problem (4.2-1) and (4.2-2) can be majorized by the solution M_i of the associated problem:

$$\partial_t M_i + \boldsymbol{\xi}_i \cdot \boldsymbol{\nabla} M_i = \frac{1}{2} \sum_{j,\,k,\,l} A_{ij}^{kl} M_k M_l, \qquad M_i(\boldsymbol{x},\,0) = K_0. \tag{4.2-8}$$

From Theorem 4.3, the functions $M_i(\boldsymbol{x},\,t)$ are independent of the space variables, and equations (4.2-8) are not partial differential equations but pure differential equations, the solution of which can next be majorized by the new associated problem

$$\frac{dL_i}{dt} = \frac{1}{2} \sup_{i,\,j,\,k,\,l} A_{ij}^{kl} \left(L_1 + L_2 + \ldots + L_b \right)^2, \qquad L_i(0) = K_0. \tag{4.2-9}$$

All the equations (4.2-9) are the same, and all the functions $L_i(t)$ are equal. We have, therefore

$$N_i(\boldsymbol{x},\,t) \leq L_i(t) = \frac{K_0}{1 - AK_0 t}. \tag{4.2-10}$$

For certain particular models, it is possible to show that, in the domain $\boldsymbol{x} \in \mathbb{R}^3$ and the interval $0 < t \leq \delta_0$ and under certain conditions on the initial values, the local solution satisfies the inequalities $0 \leq N_i(\boldsymbol{x}, t) \leq K$, where K depends only on the initial values. We can consider the instant $t = \delta_0$ as initial and repeat the argument, so the solution exists for $\delta_0 \leq t \leq \delta_0 + \delta$, with $\delta = (AK)^{-1}$. For $t = t_1 = \delta_0 + \delta$, we always have $0 \leq N_i(\boldsymbol{x}, t) \leq K$, This proves the global existence of the solution.

The simplest case in which we have such a bound K is the two-dimensional regular four velocities model, for which the discrete Boltzmann equation is

$$\partial_t N_i + c \cos\left((2i-1)\frac{\pi}{4}\right) \partial_x N_i = cS(N_{i+1}N_{i+3} - N_i N_{i+2}), \qquad i = 1,\, 2,\, 3,\, 4. \tag{4.2-11}$$

where c is a constant velocity, S a constant denoting the collisional cross section, and $N_{i+4} = N_i$; we deduce from equation (4.2-11):

$$\left.\begin{aligned} (\partial_t + c\partial_x)(N_1 + N_4) = 0, \\ (\partial_t - c\partial_x)(N_2 + N_3) = 0, \end{aligned}\right\} \tag{4.2-12}$$

$$\left.\begin{aligned} N_1(\boldsymbol{x}, t) + N_4(\boldsymbol{x}, t) = N_{10}(x - ct) + N_{40}(x - ct) \leq 2K_0, \\ N_2(\boldsymbol{x}, t) + N_3(\boldsymbol{x}, t) = N_{20}(x + ct) + N_{30}(x + ct) \leq 2K_0. \end{aligned}\right\} \tag{4.2-13}$$

As the densities are positive, they are bounded by $K = 2K_0$, and the solution exists globally. In the next section we will prove the existence of a bound K, for the three dimensional model with fourteen velocities, described in the introduction.

4.3 Global Solution for Small Initial Values

The model considered is obtained by joining the center of a cube to the vertices and to the centers of the faces. The velocities are denoted by $\boldsymbol{\zeta}_i$ $(i = 1, 2, \ldots, 8)$ and $\boldsymbol{\xi}_j$ $(j = 1, 2, \ldots, 6)$, and their components in the directions Ox, Oy, and Oz are

$$\boldsymbol{\xi}_1 = c(-1,\, 1,\, 1), \quad \boldsymbol{\xi}_2 = c(1,\, 1,\, 1), \quad \boldsymbol{\xi}_3 = c(-1,\, -1,\, 1), \quad \boldsymbol{\xi}_4 = c(1,\, -1,\, 1),$$

$$\boldsymbol{\zeta}_1 = c(1,\, 0,\, 0), \quad \boldsymbol{\zeta}_2 = c(0,\, 1,\, 0), \quad \boldsymbol{\zeta}_3 = c(0,\, 0,\, 1),$$

and $\boldsymbol{\xi}_{9-i} = -\boldsymbol{\xi}_i$ $(i = 1, 2, 3, 4)$, and $\boldsymbol{\zeta}_{j+3} = -\boldsymbol{\zeta}_j$ $(j = 1, 2, 3)$.

The velocity moduli are: $\|\boldsymbol{\zeta}_j\| = c$ and $\|\boldsymbol{\xi}_i\| = \sqrt{3}\,c$. The number density of molecules with velocity $\boldsymbol{\xi}_j$ is denoted by N_i, that of molecules with velocity $\boldsymbol{\zeta}_j$ by M_j. The nontrivial collisions, of which the post-collision velocities after differ from the pre-collision velocities, are 27 in number:

$$
\begin{array}{ll}
\text{6 collisions like the type} & (\boldsymbol{\xi}_1,\, \boldsymbol{\xi}_8) \longleftrightarrow (\boldsymbol{\xi}_2,\, \boldsymbol{\xi}_7), \\
\text{6 collisions like the type} & (\boldsymbol{\xi}_1,\, \boldsymbol{\xi}_4) \longleftrightarrow (\boldsymbol{\xi}_2,\, \boldsymbol{\xi}_5), \\
\text{3 collisions like the type} & (\boldsymbol{\zeta}_1,\, \boldsymbol{\zeta}_4) \longleftrightarrow (\boldsymbol{\zeta}_2,\, \boldsymbol{\zeta}_5), \\
\text{12 collisions like the type} & (\boldsymbol{\xi}_1,\, \boldsymbol{\zeta}_1) \longleftrightarrow (\boldsymbol{\xi}_2,\, \boldsymbol{\zeta}_4).
\end{array}
$$

In a collision we assume that a given pair of velocities give all the the possible pairs with the same probabilities, so for each of the above types the values of the probabilities are respectively 1/4, 1/2, 1/3 and 1/2; the corresponding transition probabilities are equal to the product of the above probability by the collisional cross section S and by the modulus of the relative velocity of the molecules before (or after) the collision [35], so for the above types of collision we obtain respectively, $\sqrt{3}cS/2$, $\sqrt{2}cS$, $2cS/3$, and $\sqrt{6}cS/2$. The kinetic equations (4.2-1) are obtained by writing for each density N_i (or M_j) the balance of gains and losses in molecules of velocities $\{\boldsymbol{\xi}_i\}$ (or $\{\boldsymbol{\zeta}_j\}$) during a collision. We obtain for example the two following equations

$$
\begin{aligned}
&\partial_t N_1 + c\left(-\partial_x N_1 + \partial_y N_1 + \partial_z N_1\right) \\
&\quad = \frac{\sqrt{3}}{2} cS(N_2N_7 + N_3N_6 + N_4N_5 - 3N_1N_8) \\
&\quad + \sqrt{2}cS(N_2N_3 + N_3N_5 + N_5N_2 - N_1N_4 - N_1N_6 - N_1N_7) \\
&\quad + \frac{\sqrt{6}}{2} cS(N_2M_4 + N_3M_2 + N_5M_3 - N_1M_1 - N_1M_5 - N_1M_6), \qquad (4.3\text{-}1) \\
&\partial_t M_1 + c\partial_x M_1 = \frac{3}{2} cS(M_2M_5 + M_3M_6 - 2M_1M_4) \\
&\quad + \frac{\sqrt{6}}{2} cS(N_2M_4 + N_4M_4 + N_6M_4 + N_8M_4 \\
&\quad - N_1M_1 - N_3M_1 - N_5M_1 - N_7M_1). \qquad (4.3\text{-}2)
\end{aligned}
$$

By an appropriate permutation of the indexes, we obtain additional seven equations similar to equation (4.3-1) and five equations similar to equation (4.3-2). The system of 14 equations so obtained is the discrete Boltzmann equation for the model considered. When the initial densities are independent of y and z, which we assume, so are the densities.

Therefore we look for the solution $N_i(\boldsymbol{x},\, t)$ and $M_j(\boldsymbol{x},\, t)$ which satisfies the initial conditions:

$$\left.\begin{aligned} N_i(\boldsymbol{x},\, 0) &= N_{i0}(\boldsymbol{x}), & i &= 1,\, 2,\, \ldots,\, 8, \\ M_j(\boldsymbol{x},\, 0) &= M_{j0}(\boldsymbol{x}), & j &= 1,\, 2,\, \ldots,\, 6. \end{aligned}\right\} \tag{4.3-3}$$

We assume that the initial densities are differentiable and satisfy the following conditions, in which K_0 and α_0 are two positive constants

$$0 \le N_{i0}(\boldsymbol{x}) \le K_0, \qquad 0 \le M_{j0}(\boldsymbol{x}) \le K_0, \tag{4.3-4}$$

$$\int_{-\infty}^{+\infty} \left\{ \textstyle\sum_{i=1}^{8} N_{i0}(\boldsymbol{x}) + \sum_{i=1}^{6} M_{j0}(\boldsymbol{x}) \right\} S d\boldsymbol{x} = \alpha_0. \tag{4.3-5}$$

Theorem 4.7 When the conditions (4.3-4) are satisfied, and when α_0 is less than $3/8$, the solution of the Cauchy problem defined by the discrete Boltzmann equation corresponding to the 14 velocities model and by the conditions (4.3-3) exists for all $\boldsymbol{x}$ and for all positive t.

The local solution exists in the interval $0 < t \le \delta_0 = (AK_0)^{-1}$, where $A = 196\sqrt{2}cS$. To prove the global existence it is sufficient to prove the existence of a positive bound K, so that for all $\boldsymbol{x}$ and for $0 < t \le \delta_0$, we have

$$N_i(\boldsymbol{x},\, t) \le K, \qquad M_i(\boldsymbol{x},\, t) \le K. \tag{4.3-6}$$

To prove the existence of such a bound we consider the sums of the densities of the velocities having the same components on the x-axis, we put

$$\left.\begin{aligned} A_1 &= N_2 + N_4 + N_6 + N_8 + M_1, && \text{(with } +c \text{ } x\text{-component)}, \\ A_2 &= N_1 + N_3 + N_5 + N_7 + M_4, && \text{(with } -c \text{ } x\text{-component)}, \\ 2A_3 &= M_2 + M_3 + M_5 + M_6. && \text{(with } 0 \text{ } x\text{-component)}. \end{aligned}\right\} \tag{4.3-7}$$

From the kinetic equations (4.3-1), (4.3-2), and the other similar ones, we deduce:

$$\left.\begin{aligned} &\partial_t A_i + \xi_i \partial_x A_i = f_i(x,\, t), \qquad i = 1,\, 2,\, 3, \\ &f_1 = f_2 = -f_3 = \frac{2}{3} cS(M_2M_5 + M_3M_6 - 2M_1M_4), \end{aligned}\right\} \tag{4.3-8}$$

where $\xi_1 = c$, $\xi_2 = -c$ and $\xi_3 = 0$. As the initial densities are positive or zero, so are the densities A_i and a bound for the functions $A_i(x,\, t)$ is a bound for the densities. Equations (4.3-8) can be integrated in the form

$$A_i(x,\, t) = A_i(x - \xi_i t,\, 0) + \int_0^t f_i(x - \xi_i s,\, t - s)\, ds, \qquad i = 1,\, 2,\, 3. \tag{4.3-9}$$

The functions $A_i(x-\xi_i t,\, 0)$ are bounded by $5K_0$, and if we denote by K a bound for the densities in the domain $x \in \mathbb{R}$, $0 < t \le \delta_0$, then the integrals are bounded by one of the following expressions:

$$\frac{2}{3} cSK \int_0^t (M_2 + M_3)(x - \xi_i s,\, t - s)\, ds, \qquad i = 1,\, 2, \tag{4.3-10}$$

$$\frac{4}{3} cSK \int_0^t M_1(x,\, t-s)\, ds. \tag{4.3-11}$$

Following the method of Nishida and Mimura [55], it is possible to majorize the integrals given in (4.3-10) and (4.3-11) by integrating the conservation equations

$$\partial_t (A_i + A_3) + \xi_i \partial_x A_i = 0, \qquad i = 1,\, 2, \tag{4.3-12}$$

over triangles AA_iA_3 of the x-t plane. The points A and A_i have as coordinates $(x,\, t)$ and $(x - \xi_i t,\, 0)$, respectively. We have also as a consequence of relations (4.3-12):

$$\partial_t (A_1 + A_2 + 2A_3) + c\partial_x (A_1 - A_2) = 0, \tag{4.3-13}$$

which we integrate over the triangle AA_1A_2. Stokes' theorem gives

$$\begin{aligned} &2\int_0^t (A_2 + A_3)(x - cs,\, t-s) c\, ds + 2\int_0^t (A_2 + A_3)(x + cs,\, t-s) c\, ds \\ &\qquad = \int_{x-ct}^{x+ct} \{A_1(x,\, 0) + A_2(x,\, 0) + A_3(x,\, 0)\}\; dx \le \frac{\alpha_0}{S} \end{aligned} \tag{4.3-14}$$

$$\begin{aligned} &\int_0^t A_1(x,\, t-s) c\, ds + \int_0^t A_3(x - cs,\, t-s) c\, ds \\ &\qquad = \int_{x-ct}^{t} \{A_1(x,\, 0) + A_3(x,\, 0)\}\; dx \le \frac{\alpha_0}{S}. \end{aligned} \tag{4.3-15}$$

From formula (4.3-14) we deduce that the integral (4.3-10) is less than $\frac{2}{3}\alpha_0 K$, and from formula (4.3-15) that the integral (4.3-11) is less than $\frac{4}{3}\alpha_0 K$. As a consequence, the formulae (4.3-9) gives

$$\left.\begin{aligned} &\sup_{x,\,t} A_i(x,\, t) \le 5K_0 + \frac{2}{3}\alpha_0 K, \qquad i = 1,\, 2, \\ &\sup_{x,\,t} A_3(x,\, t) \le 2K_0 + \frac{4}{3}\alpha_0 K, \end{aligned}\right\} \tag{4.3-16}$$

or

$$K \le 2 \sup_{x,\,t} A_i \le 10K_0 + \frac{8}{3}\alpha_0 K, \tag{4.3-17}$$

$$K \le \frac{10K_0}{1 - \frac{8}{3}\alpha_0}, \qquad \text{if } \alpha_0 \le \frac{3}{8}, \tag{4.3-18}$$

which proves Theorem 4.7

4.4 Global Solution for Periodic Initial Values

The global existence theorem proved in the previous Section assumes that the initial mass in a tube of cross-section S is sufficiently small. For a plane regular model with four velocities, Crandall and Tartar [62], using the H-Boltzmann theorem, have been able to drop this assumption when the initial densities $\{N_{i0}(\boldsymbol{x})\}$ depend only on x and are periodic. The demonstration of Tartar and Crandall is valid for all models for which the results of the previous section are valid: existence of a bound of the local solution.

The initial densities being independent of y and z, so are the densities $\{N_i(\boldsymbol{x},\, t)\}$ which are periodic in x; we will denote the period by ϖ. We have therefore to solve the following problem

$$\partial_t N_i + \xi_i \partial_x N_i = \frac{1}{2}\sum_{j,\,k,\,l} A_{ij}^{kl}(N_k N_l - N_i N_j), \qquad \text{for} \quad i = 1,\, 2,\, \ldots,\, b, \tag{4.4-1}$$

$$N_i(x,\, 0) = N_{i0}(x), \tag{4.4-2}$$

with $0 \leq N_{i0}(x) \leq K_0$, and $N_{i0}(x+\varpi) = N_{i0}(x)$.

By multiplying the two sides of Eq. (4.4-1) by $(1+\ln N_i)$ and by adding the equations obtained for all values of i, we obtain

$$\sum_{i=1}^{b}(\partial_t + \xi_i\partial_x)(N_i \ln N_i) = \frac{1}{2}\sum_{j,\,k,\,l} A_{ij}^{kl} \ln\left(\frac{N_i N_j}{N_k N_l}\right)(N_k N_l - N_i N_j). \tag{4.4-3}$$

The right hand side is negative or zero, so therefore is the left hand side, and so is its integral over an arbitrary interval, in particular over a period. But

$$\int_0^{\varpi}\frac{\partial}{\partial t}(N_i \ln N_i)\, dx = \frac{d}{dt}\int_0^{\varpi} N_i \ln N_i\, dx, \qquad \int_0^{\varpi}\frac{\partial}{\partial x}(N_i \ln N_i)\, dx = N_i \ln N_i\Big|_0^{\varpi} = 0. \tag{4.4-4}$$

We conclude that

$$\frac{d}{dt}\int_0^{\varpi}\left(\textstyle\sum_{i=1}^{b} N_i \ln N_i\right)\, dx \leq 0. \tag{4.4-5}$$

The sum of the second part in the right-hand side of equations (4.4-1) is zero, and the formulae (4.4-4) are always true if we replace $N_i \ln N_i$ by N_i; thus we have

$$\frac{d}{dt}\int_0^{\varpi}\left(\textstyle\sum_{i=1}^{b} N_i(x,\, t)\right)\, dx = 0. \tag{4.4-6}$$

and

$$\frac{d}{dt}\int_0^{\varpi}\left\{\textstyle\sum_{i=1}^{b} N_i \ln(N_i/K_0)\right\}\, dx \leq 0. \tag{4.4-7}$$

The function

$$I(t) = \int_0^{\varpi} \left\{ \sum_{i=1}^b N_i \ln(N_i/K_0) \right\} \, dx \tag{4.4-8}$$

is decreasing and negative for $t = 0$, so for all positive values of the time, $I(t)$ is negative. From the inequality

$$x|\ln x| \leq x \ln x + \frac{2}{e}, \tag{4.4-9}$$

we deduce that

$$\sum_{i=1}^b \int_0^{\varpi} N_i |\ln(N_i/K_0)| \, dx \leq \frac{2}{e} b\varpi K_0 < b\varpi K_0. \tag{4.4-10}$$

This last inequality allows us to over-estimate the integral

$$J = \sum_{i=1}^b \int_{x-cT}^{x+cT} N_i(x,\, t) \, dx, \tag{4.4-11}$$

in which the interval of integration $2cT$ is positive and less than the period ϖ. To obtain such a bound we divide the integral J into two parts, J_1 and J_2:

$$\begin{aligned} J_1 : &\qquad 0 \leq N_i(x,\, t) \leq K_0 e^m, \\ J_2 : &\qquad K_0 e^m \leq N_i(x,\, t), \end{aligned}$$

where m is a positive number. Of course J_1 is smaller than $b2cTK_0e^m$, and for J_2 we have

$$N_i \leq \frac{N_i}{m} \ln \frac{N_i}{K_0} = \frac{N_i}{m} \left| \ln \frac{N_i}{K_0} \right|, \tag{4.4-12}$$

and therefore

$$J_2 \leq \frac{1}{m} \sum_{i=1}^b \int_0^{\varpi} N_i \left| \ln \frac{N_i}{K_0} \right| \, dx \leq \frac{b\varpi}{m} K_0, \tag{4.4-13}$$

$$J \leq b\varpi K_0 \left(\alpha e^m + \frac{1}{m} \right), \qquad \text{with} \quad \alpha = \frac{2cT}{\varpi}. \tag{4.4-14}$$

When α, positive, is fixed, the function of m, $\alpha e^m + \frac{1}{m}$ has a minimum equal to $(m+1)/m^2$ on the infinite interval $m > 0$, reached for $\alpha e^m = m^{-2}$. The function

$$\frac{4m^2}{m+1} - (1 + m + 2\ln m), \qquad m > 0, \tag{4.4-15}$$

has a minimum for $m = 1$, and this minimum is zero; therefore for $\alpha < e$ (or $m > 0.47767$)

$$\frac{m+1}{m^2} \leq \frac{4}{1 + m + 2\ln m} = \frac{4}{1 - \ln \alpha}. \tag{4.4-16}$$

As m is arbitrary we choose the value which corresponds to the minimum (the root of the equation $\alpha m^2 e^m - 1 = 0$) and we obtain

$$\sum_{i=1}^{b} \int_{x-cT}^{x+cT} N_i(x,\, t)\, dx < \frac{4\varpi b K_0}{1 - \ln \alpha}, \qquad \alpha = \frac{2cT}{\varpi}. \tag{4.4-17}$$

Returning to the 14 velocities model, we denote by $N_i(x,\, t)$ the densities (i varies from 1 to 14), and we consider the functions $\{\widetilde{N}_i(x,\, t)\}$ which satisfy the following conditions

$$\left.\begin{array}{ll} \widetilde{N}_i(x,\, t_1) = N_i(x,\, t_1), & |x - X| \le cT, \\ \widetilde{N}_i(x,\, t_1) = 0, & |x - X| > cT. \end{array}\right\} \tag{4.4-18}$$

For all positive values t_1 of the time, the functions $\{\widetilde{N}_i(x,\, t_1)\}$ satisfy the relation

$$\int_{-\infty}^{+\infty} \left(\textstyle\sum_{i=1}^{14} \widetilde{N}_i(x,\, t_1) \right) S\, dx < \alpha_1 = \frac{56\varpi S K_0}{1 - \ln \frac{2cT}{\varpi}}. \tag{4.4-19}$$

If we choose

$$\frac{2cT}{\varpi} \le \exp\left(1 - \frac{448}{3}\varpi S K_0\right), \tag{4.4-20}$$

the quantity α_1 is less than 3/8 and we can apply Theorem 4.7: the functions $\widetilde{N}_1(x,\, t)$ exist for all values of $x \in \mathbb{R}$ and $t > t_1$. In the triangle with vertices $(X,\, t_1 + T)$ and $(X \pm cT,\, t_1)$ the solution $\widetilde{N}_1(x,\, t)$ coincides with the solution of the kinetic equations which takes the values $N_i(x,\, t_1)$ for $t = t_1$: as X is arbitrary, this proves the existence of the solution for $t_1 < t \le t_1 + T$. The inequality (4.4-19) is still valid for $t_2 = t_1 + T$, hence existence also holds for $t_1 + T < t \le t_1 + 2T$; the argument can be repeated, and as t_1 is arbitrary and can be chosen less than δ_0, the global existence follows.

Using now the inequality (4.3-18) we have (see the details of the proof in Ref. [14]):

$$\sup_{x,\,t} N_i(x,\, t) \le \frac{10K_1}{1 - \frac{8}{3}\alpha_1}, \qquad x \in \mathbb{R}, \quad t_1 < t \le t_1 + T, \tag{4.4-21}$$

$$K_1 = \sup_{x,\,t} N_i(x,\, t), \qquad x \in \mathbb{R}. \tag{4.4-22}$$

As the inequality (4.4-19) is valid for arbitrary positive values of t_1, we can also write

$$\sup_{x,\,t} N_i(x,\, t) \le \left(\frac{10}{1 - \frac{8}{3}\alpha_1} \right)^n K_1, \quad \forall\, x \in \mathbb{R},\ t_1 + (n-1)T < t \le t_1 + nT. \tag{4.4-23}$$

$$N_i(x,\, t) \le K_2 e^{\lambda t}, \qquad \forall\, x \in \mathbb{R}, \tag{4.4-24}$$

with appropriate choice of K_2 and λ. The inequality (4.4-23) is obtained from (4.4-21) iteratively for intervals $(n-1)T < t - t_1 \le nT$. Consequently the suppression of the condition $\alpha_0 < 3/8$ and its replacement by the condition of periodicity of the initial values has as a consequence that we cannot conclude that the densities are bounded; however it is possible to majorize them by an exponential function.

4.5 Global Solution for Bounded Initial Values

When the initial densities $\{N_{i0}(x)\}$ satisfy only the conditions $0 \leq N_{i0}(x) \leq K_0$, it is possible to define new initial values $\{\tilde{N}_{i0}(x)\}$ periodic, with period ϖ, continuously differentiable and satisfying the conditions

$$\tilde{N}_{i0}(x) = N_{i0}(x), \qquad \text{for } |x - X| \leq cT < \frac{\varpi}{2}. \tag{4.5-1}$$

The functions $\{\tilde{N}_{i0}(x,t)\}$ corresponding to these initial values exist in an arbitrary large area of the x-t plane and coincide with the solution corresponding to the initial values $\{\tilde{N}_{i0}(x)\}$ in the triangle with vertices $(X,\, t)$ and $(X \pm cT,\, 0)$ on the X-$(x-t)$ plane. As X is arbitrary the solution $N_i(x,\, t)$ exists for $x \in \mathbb{R},\ 0 < t \leq T$; but as T is also arbitrary, it exists for all x and all positive t. Thus we have

Theorem 4.8 If the initial densities are continuous, differentiable and bounded, the solution of the initial value problem, for the fourteen velocity model, exists globally.

4.6 Case of the Plane Regular Model

Another model for which a global existence can be proved is the plane regular model with $2r$ velocities $\boldsymbol{\xi}_k$, the components of the velocity being

$$u_k = c\cos\left(\frac{2k-1}{2r}\pi\right), \quad v_k = c\sin\left(\frac{2k-1}{2r}\pi\right), \quad w_k = 0. \tag{4.6-1}$$

Denoting by N_k the density of molecules with velocity $\boldsymbol{\xi}_k$, and considering that $N_j = N_i$ for $j = i \pmod{2r}$, we can write the kinetic equation in the following form [13]:

$$\left.\begin{aligned} &\partial_t N_k + \boldsymbol{\xi}_k \cdot \boldsymbol{\nabla} N_k = f_k, \quad k = 1,\, 2,\, \ldots,\, 2r, \\ &f_k = \frac{2cS}{r}\left\{\sum_{l=k+1}^{k+r}(N_l N_{l+r} - N_k N_{k+r})\right\}, \end{aligned}\right\} \tag{4.6-2}$$

where c and S are two positive constants.

We are always looking for the solutions $N_k(\boldsymbol{x},\, t)$ satisfying the initial conditions

$$N_k(\boldsymbol{x},\, 0) = N_{k0}(x), \quad k = 1,\, 2,\, \ldots,\, 2r, \tag{4.6-3}$$

with

$$0 \leq N_{k0}(x) \leq K_0. \tag{4.6-4}$$

$$\int_{-\infty}^{+\infty} \left\{\textstyle\sum_{k=1}^{2r} N_{k0}(x)\right\} S\, dx = \alpha_0, \tag{4.6-5}$$

where K_0 and α_0 are two positive constants, the same as in Section 4.3.

Theorem 4.9 When the the conditions (4.6-4) and (4.6-5) are satisfied and if $2\alpha_0 < 1$, (4.6-2) and (4.6-3) exists for all $\boldsymbol{x}$, and all positive t.

Of course from Theorem 4.3 the densities depend only on x and t, and the local solution exists in the domain (Δ): $x \in \mathbb{R}$, $0 < t \leq \delta_0$. To prove the global existence, it is sufficient to prove that in (Δ) the densities $\{N_k(\boldsymbol{x},\, t)\}$ are bounded by a number K independent of δ_0. Equation (4.6-2) can be integrated in the form

$$N_k(x,\, t) = N_{k0}(x - u_k t) + \int_0^t f_k(x - u_k s,\, t - s)\, ds. \tag{4.6-6}$$

Putting

$$A_m(x,\, t) = \sum_{l=m+1}^{m+r} N_l(x,\, t), \qquad B_m(x,\, t) = \sum_{l=m+1}^{m+r} u_l N_l(x,\, t), \tag{4.6-7}$$

and denoting by K the upper bound of the densities $\{N_k(\boldsymbol{x},\, t)\}$ in the domain (Δ), we can majorized the second member of equation (4.6-6) by:

$$K_0 + \frac{2}{r} cSK \int_0^t A_m(x - u_k s,\, t - s)\, ds, \tag{4.6-8}$$

where A_m can be replaced by $(A_m - N_k)$ if $m < k \leq m+r$, or by $(A_m - N_{k+r})$ if $m - r < k < m$. The integral as in Section 4.3 can be majorized by integration of the following conservation equations

$$\partial_t A_m + \partial_x B_m = 0 \tag{4.6-9}$$

on the triangles of the x-t plane with vertices $A := (x, t)$ and $A_i := (x - u_i t,\, 0)$:

$$\iint_{AA_iA_j} (\partial_t A_m + \partial_x B_m)\, dxdt = 0. \tag{4.6-10}$$

Then Stokes' theorem gives us

$$\int_0^t (u_i A_m - B_m)(x - u_i s,\, t-s)\, ds + \int_0^t (B_m - u_j A_m)(x - u_j s,\, t-s)\, ds$$
$$= \int_{x-u_i t}^{x-u_j t} A_m(x,\, 0)\, dx \le \frac{\alpha_0}{S}. \tag{4.6-11}$$

By choosing $u_j = u_r$ and $u_i = u_k$, and $u_i = u_l$ and $u_j = u_k$, we deduce, respectively, the following two inequalities:

$$\int_0^t (u_i A_m - B_m)(x - u_i s,\, t-s)\, ds \le \alpha_0, \tag{4.6-12}$$
$$\int_0^t (B_m - u_j A_m)(x - u_j s,\, t-s)\, ds \le \alpha_0. \tag{4.6-13}$$

Now if r is even, *i.e.* $r = 2q$, u_k is never zero and we have furthermore $B_q > 0$, $B_{3q} < 0$ and $|u_k| > u_q$, therefore

$$\left.\begin{aligned} u_k A_q - B_q > u_q A_q = c \sin\left(\frac{\pi}{2r}\right) A_q, \qquad & \text{if } u_k > 0, \\ B_{3q} - u_k A_{3q} > u_q A_{3q} = c \sin\left(\frac{\pi}{2r}\right) A_{3q}, \qquad & \text{if } u_k < 0. \end{aligned}\right\} \tag{4.6-14}$$

It is also always possible to choose m ($m = q$ if $u_k > 0$, and $m = 3q$ if $u_k < 0$) so that the integral in the formula (4.6-8) can be majorized by

$$\frac{2}{r} cSK \frac{\alpha_0}{cS \sin(\pi/2r)} \le 2K\alpha_0. \tag{4.6-15}$$

If r is odd, $r = 1 + 2q$, u_k is zero for $k = 1 + q$ and $k = 2 + 3q$. For $u_k \neq 0$, we obtain the majorization (4.6-15) and for $k = 1 + q$ we have

$$u_k A_k - B_k = -B_k > c \sin\left(\frac{\pi}{2r}\right)(A_k - N_{k+r}), \tag{4.6-16}$$

and we also obtain the bound given by (4.6-15) if in the formula we replace A_m by $(A_m - N_{k+r})$. Finally we can write

$$N = \sup_{(x,t)\in(\Delta)} N_k(x,\, t) \le K_0 + 2\alpha_0 K, \tag{4.6-17}$$
$$K \le \frac{K_0}{1 - 2\alpha_0}, \quad \text{if } 2\alpha_0 < 1, \tag{4.6-18}$$

which proves the global existence of the solution of the initial value problem, when α_0 is small enough. Then we can prove the global existence for periodic initial values and for bounded initial values as in Sections 4.4 and 4.5. The result is valid for all finite values of r, when r increases indefinitely the densities $\{N_k(\boldsymbol{x},\, t)\}$ are to be replaced by a unique density $N(x,\, \theta,\, t)$ which depends on the abscissa x, on the direction θ of the velocity $\boldsymbol{\xi}$, and on the time t. The limiting forms of equations (4.6-1) and (4.6-2) are

$$\partial_t N + c\cos\theta\,\partial_x N = \frac{cS}{\pi}\int_0^{2\pi}\{N(\phi)N(\phi+\pi) - N(\theta)N(\theta+\pi)\}\,d\phi, \tag{4.6-19}$$

$$N(\boldsymbol{x},\, \theta,\, 0) = N_0(x,\, \theta). \tag{4.6-20}$$

The equation (4.6-19) is called the semi-discrete Boltzmann equation (SDBE), because the velocities are discretized in modulus, not in direction.

4.7 Conclusion

The existence of global solution for the initial value problem has been proved when the initial densities are given on the entire real axis. The same conditions in the existence of global solution also appear in the shock tube problem when the tube is unbounded in both directions. It is possible to prove the global existence of the solution in the case of a tube which is either semi-infinite or bounded at both ends [14]. Also it is possible to consider models with a larger number of velocities. Based on the model with 14 velocities we can construct a model with 20 velocities by adding velocities equal to twice the median velocities (velocities $\{\boldsymbol{\zeta}_j\}$); in this model the velocities have 3 different moduli, and there are 42 nontrivial collisions. Then by addition of velocities equal to twice the diagonal velocities (velocities $\{\boldsymbol{\xi}_i\}$) we obtain a model with 28 velocities, and 4 different moduli in which there are 66 nontrivial collisions. For these models, probably, the global existence of the initial value problem can be proved. The process can be repeated indefinitely, we obtain models with $(14n-6)$, $14n$, or $(14n+6)$ velocities. In the case of $14n$ velocities, for example, the number of nontrivial collisions is $[27n + 12(n-1)]$, and as this number increases with n, we can expect to have a better approximation of the exact Boltzmann equation. It would be interesting to extend the results to the semi-discrete Boltzmann equation, *i.e.*, Eq. (4.6-19), which is an integro-partial differential equation, and for this reason more similar to the original Boltzmann equation, but for the moment this is an open problem.

4.8 Some Recent Developments

Since 1980, when the Lecture Notes were written, much work has been done and published on the subject of global solutions of the discrete Boltzmann equation. This Section provides a brief summary of some recent results.

First, Cornille [30] and Bobylev [5] obtained analytic solutions for the Broadwell model with six velocities [10]. Based on Cornille's work [30], Cabannes and Tiem obtained exact solutions for models with speeds of different moduli [26]. Cabannes and Duruisseau obtained similar exact solutions by using symbolic computational software Macsyma [22].

Second, much progress has been made since the first result by Nishida and Mimura [55] on the existence of global solutions of the Broadwell model, with sufficiently small initial data, when time goes to infinity. Bony considered the existence of global solutions for the general model in one dimension [8, 9]. For the Broadwell model with bounded initial data, Tartar proved existence of global solutions [62, 63], and Cabannes extended this result to more general models [13, 14, 15, 17]. Balabane [2] and Cabannes [16] studied the Carleman model [27] and the Broadwell model [10], respectively, with partially negative initial data. Beale and Alvès studied the behavior of global solutions as time goes to infinity for the Broadwell model [3] and a model with 14 velocities [1], respectively. Cabannes studied the same problem for models including triple collisions [18].

Third, in his Ph.D. thesis [53], Mischler first studied the convergence of the solutions of the discrete velocity models to the solutions of the corresponding Boltzmann equations when the number of the discrete velocities is infinite, *i.e.*, when the discrete velocity set is a D-dimensional lattice space. Mischler's study marks the starting point of many subsequent works [7, 54, 57, 56, 51]. A key point in the proof of convergence is to understand distributions of lattice points on a given sphere, and the only regular distributions of lattice points on a sphere are those related to the five regular polyhedrons. However, Pałczewski and Schneider have shown that it is necessary, and possible, to define distributions which are "*almost*" regular, and the number of points is as large as one wants [56].

Finally, Cabannes *et al.* have been able to obtain exact solutions for the semi-continuous Boltzmann equation (SCBE) [64, 26, 24, 25, 23]. When the functions of velocities have period π in θ, where θ is the polar angle of the velocity in a plane [cf. Eq. (4.6-19)], Sibgatullin and Cabannes obtained the general solution of the SCBE in parametric form [59]. Based on the knowledge of the general solutions of SCBE, it is also possible to investigate the existence of "eternal" solutions, *i.e.*, solutions which exist for all times, future and the past. It is conjectured that the only eternal solutions of the original Boltzmann equation are the solutions of Maxwell. This conjecture is known as the conjecture of the positive eternal solutions. The conjecture has not been proved except for some simple model equations, and so far the best result for the general case is due to Villani [65]. The conjecture has been proved by Cabannes [19, 20, 21] for the case of SCBE when the initial data, hence the solutions, have a period π in θ. The proof is based on the knowledge of the general solution in parametric form [59].

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